Digital Receivers

Channel: adds noise to the signal
(Additive White Gaussian Noise)

Receiver: seeks to determine bits in $s(t)$ from the noisy received signal $r(t)$.

Metric: Probability of error.
   i.e., Probability that $\hat{b}_n \neq b_n$

- Return to receiver design and analysis on Wednesday
- Today: Absolutely, most essential probability results.
Error probability (on a binary symmetric channel)

**Binary Symmetric Channel (BSC)**

Transmitted  

\[
\begin{align*}
0 & \rightarrow 0 \\
1 & \rightarrow 1 \\
0 & \rightarrow 1 \\
1 & \rightarrow 0
\end{align*}
\]

\[ \begin{array}{c}
0 \\
\text{a} \\
b \\
1
\end{array} \]

\[ \begin{array}{c}
1-a \\
\text{p} \\
1-b \\
\end{array} \]

Also:
- 0 and 1 are sent with a prior probabilities \( \pi_0, \pi_1 \), respectively.

BSC is characterized by transition probabilities:
- Conditional probabilities
  \[ \Pr[R = r | T = t] \]
- E.g.: \( \Pr[R = 0 | T = 0] = 1 - a \)

Q: What is the probability of error?

\[ P_e = \Pr[R \neq T] \]

Law of Total Prob.

\[ P_e = \pi_1 \cdot \Pr[R = 0 | T = 1] + \pi_0 \cdot \Pr[R = 1 | T = 0] \]

- We will always compute \( P_e \) this way
- Transition probabilities depend on
  - Signal constellation
  - Pulse shape
  - Noise
  - Receivers structure
Facts about Gaussian random variables

**Notation:** \( X \sim N(\mu, \sigma^2) \)

**Means:** random variable \( X \) is Gaussian (Normal) with mean \( \mu \) and variance \( \sigma^2 \).

\( X \) has a probability density function (pdf):

\[
    f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
\]

**Two parameters:**

- **Mean:** \( E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx = \mu \)
- **Variance:** \( E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x) \, dx = \sigma^2 \)

**Cumulative distribution function (cdf)**

- \( F_X(x) = \int_{-\infty}^{x} f_X(z) \, dz \)
  - Often need:
  - \( \Pr(X > \delta) = \int_{\delta}^{\infty} f_X(x) \, dx \)
  - \( = \int_{\delta}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx \)
To simplify this integral:

\[
Z = \frac{x - \mu}{\sigma}, \quad dZ = \frac{dx}{\sigma}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot dZ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} \cdot dz \equiv Q\left(\frac{\mu - \mu}{\frac{\sigma}{\sqrt{2}}}\right)
\]

where

\[
Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot dx
\]

**Important Property of Gaussians**

The sum of Gaussian random variables is itself a Gaussian random variable.

\[\Rightarrow \text{ only need to find mean & variance}\]

**Example:** Let \(X_1, X_2, \ldots, X_n\) be independent Gaussians: \(X_n \sim N(\mu, \sigma^2)\)

Then:
\[Y = \sum_{n=1}^{N} X_n \sim N\]

Mean:
\[E[Y] = E\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{N} E[X_n] = N \cdot \mu\]
Variance:

\[
\text{Var}[Y] = \mathbb{E}[(Y - \mathbb{E}[Y])^2]
\]

\[
= \mathbb{E}[(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \mu)^2]
\]

\[
= \mathbb{E}\left(\sum_{n=1}^{N} (x_n - \mu)^2\right)
\]

\[
= \mathbb{E}\left[\sum_{n=1}^{N} \sum_{m=1}^{N} (x_n - \mu)(x_m - \mu)\right]
\]

\[
= \sum_{n=1}^{N} \sum_{m=1}^{N} \mathbb{E}[x_n - \mu \cdot (x_m - \mu)]
\]

\[
\Rightarrow \begin{cases} 
6^2 & \text{if } n=m \\
0 & \text{if } n \neq m 
\end{cases}
\]

\[
n \neq m: \mathbb{E}[x_n - \mu \cdot (x_m - \mu)]
\]

\[
\text{independence} = \mathbb{E}[(x_n - \mu) \cdot \mathbb{E}(x_m - \mu)]
\]

\[
\text{dependence} = (\mathbb{E}[x_n - \mu]) \cdot (\mathbb{E}[x_m - \mu]) = 0
\]

\[
\Rightarrow \text{Var}[Y] = N \cdot 6^2
\]

\[
\Rightarrow Y \sim N(N \mu, N \cdot 6^2)
\]