ECE 201: Introduction to Signal Analysis

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Part I

Introduction
Lecture: Introduction
Learning Objectives

➤ Intro to Electrical Engineering via Digital Signal Processing.
➤ Develop initial understanding of Signals and Systems.
➤ Learn MATLAB
➤ Note: Math is not very hard - just algebra.
DSP - Digital Signal Processing

Digital: processing via computers and digital hardware we will use PC’s.

Signal: Principally signals are just functions of time
- Entertainment/music
- Communications
- Medical, . . .

Processing: analysis and transformation of signals we will use MATLAB
Outline of Topics

- Sinusoidal Signals
- Time and Frequency representation of signals
- Sampling
- Filtering

- MATLAB
  - Lectures
  - Labs
  - Homework
Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expressed as sums of sinusoids (Fourier Theory)
- Bridge to frequency domain.
- Sinusoids are special signals for linear filters (eigenfunctions).
- Manipulating sinusoids is much easier with the help of complex numbers.
Time and Frequency

- Closely related via sinusoids.
- Provide two different perspectives on signals.
- Many operations are easier to understand in frequency domain.
Sampling

- Conversion from continuous time to discrete time.
- Required for Digital Signal Processing.
- Converts a signal to a sequence of numbers (samples).
- Straightforward operation
  - with a few *strange* effects.
Filtering

- A simple, but powerful, class of operations on signals.
- Filtering transforms an *input signal* into a more suitable *output signal*.
- Often best understood in frequency domain.
Relationship to other ECE Courses

- Next steps after ECE 201:
  - ECE 220/320: Signals and Systems
  - ECE 285/286: Circuits
- Core courses in controls and communications:
  - ECE 421: Controls
  - ECE 460: Communications
- Electives:
  - ECE 410: DSP
  - ECE 450: Robotics
  - ECE 463: Digital Comms
Part II

Sinusoids and Complex Exponentials
Lecture: Introduction to Sinusoids
The Formula for Sinusoidal Signals

- The general formula for a sinusoidal signal is

\[ x(t) = A \cdot \cos(2\pi ft + \phi). \]

- \( A, f, \) and \( \phi \) are parameters that characterize the sinusoidal signal.
  - \( A \) - Amplitude: determines the height of the sinusoid.
  - \( f \) - Frequency: determines the number of cycles per second.
  - \( \phi \) - Phase: determines the location of the sinusoid.
The formula for this sinusoid is:

\[ x(t) = 3 \cdot \cos(2\pi \cdot 400 \cdot t + \pi/4). \]
The Significance of Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expressed as sums of sinusoids (Fourier Theory)
  - Provides bridge to frequency domain.
- Sinusoids are *special signals* for linear filters (eigenfunctions).
- Sinusoids occur naturally in many situations.
  - They are solutions of differential equations of the form
    \[ \frac{d^2 x(t)}{dt^2} + ax(t) = 0. \]
- Much more on these points as we proceed.
The properties of sinusoidal signals stem from the properties of the cosine function:

- **Periodicity:** \( \cos(x + 2\pi) = \cos(x) \)
- **Eveness:** \( \cos(-x) = \cos(x) \)
- **Ones of cosine:** \( \cos(2\pi k) = 1 \), for all integers \( k \).
- **Minus ones of cosine:** \( \cos(\pi(2k + 1)) = -1 \), for all integers \( k \).
- **Zeros of cosine:** \( \cos\left(\frac{\pi}{2}(2k + 1)\right) = 0 \), for all integers \( k \).
- **Relationship to sine function:** \( \sin(x) = \cos(x - \pi/2) \) and \( \cos(x) = \sin(x + \pi/2) \).
Amplitude

- The amplitude $A$ is a *scaling factor*.
- It determines how large the signal is.
- Specifically, the sinusoid oscillates between $+A$ and $-A$. 
Frequency and Period

- Sinusoids are **periodic** signals.
- The frequency \( f \) indicates how many times the sinusoid repeats per second.
- The duration of each cycle is called the **period** of the sinusoid. It is denoted by \( T \).
- The relationship between frequency and period is

\[
f = \frac{1}{T} \quad \text{and} \quad T = \frac{1}{f}.
\]
Phase and Delay

- The phase $\phi$ causes a sinusoid to be shifted sideways.
- A sinusoid with phase $\phi = 0$ has a maximum at $t = 0$.
- A sinusoid that has a maximum at $t = t_1$ can be written as

$$x(t) = A \cdot \cos(2\pi f(t - t_1)).$$

- Expanding the argument of the cosine leads to

$$x(t) = A \cdot \cos(2\pi ft - 2\pi ft_1).$$

- Comparing to the general formula for a sinusoid reveals

$$\phi = -2\pi ft_1 \text{ and } t_1 = \frac{-\phi}{2\pi f}.$$
Sinusoidal Signals

Complex Exponential Signals

$T = \frac{1}{f}$

Time (s)

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ECE 201: Intro to Signal Analysis
Exercise

1. Plot the sinusoid

\[ x(t) = 2 \cos(2\pi \cdot 10 \cdot t + \pi/2) \]

between \( t = -0.1 \) and \( t = 0.2 \).

2. Find the equation for the sinusoid in the following plot
Exercise

- The sinusoid below has frequency $f = 10$ Hz.
- Three of its maxima are at the following locations
  $t_1 = -0.075$ s, $t_2 = 0.025$ s, $t_3 = 0.125$ s
- Use each of these delays to compute a value for the phase $\phi$ via the relationship $\phi_i = -2\pi ft_i$.
- What is the relationship between the phase values $\phi_i$ you obtain?
Lecture: Continuous-time and Discrete-Time Signals
So far, we have been referring to sinusoids of the form

\[ x(t) = A \cdot \cos(2\pi ft + \phi). \]

Here, the independent variable \( t \) is continuous, i.e., it takes on a continuum of values.

Signals that are functions of a continuous time variable \( t \) are called continuous-time signals or, sometimes, analog signals.
Sampling and Discrete-Time Signals

- MATLAB, and other digital processing systems, can not process continuous-time signals.
- Instead, MATLAB requires the continuous-time signal to be converted into a discrete-time signal.
- The conversion process is called sampling.
- To sample a continuous-time signal, we evaluate it at a discrete set of times $t_n = nT_s$, where
  - $n$ is an integer,
  - $T_s$ is called the sampling period (time between samples),
  - $f_s = 1 / T_s$ is the sampling rate (samples per second).
- In MATLAB, the set of sampling times $t_n$ is usually defined by a command like:

```matlab
% sampling times between 0 and 5 with sampling period Ts
tt = 0 : Ts : 5;
```
Sampling and Discrete-Time Signals

- Sampling means evaluating $x(t)$ at time instances $nT_s$ and results in a sequence of samples

$$x(nT_s) = A \cdot \cos(2\pi fnT_s + \phi).$$

- Note that the independent variable is now $n$, not $t$.
- To emphasize that this is a discrete-time signal, we write

$$x[n] = A \cdot \cos(2\pi fnT_s + \phi).$$

- Sampling is a straightforward operation.
- But the sampling rate $f_s$ must be chosen with care!
Vectors and Matrices

MATLAB is specialized to work with vectors and matrices.
Most MATLAB commands take vectors or matrices as arguments and perform looping operations automatically.
Creating vectors in MATLAB:

directly:

\[ x = [1, 2, 3]; \]

using the increment (:) operator:

\[ x = 1:2:10; \]

produces a vector with elements \([1, 3, 5, 7, 9]\).

using MATLAB commands  For example, to read a .wav file

\[ [x, fs] = audioread(‘music.wav’); \]
Plot a Sinusoid

```matlab
%% parameters
A = 3;
f = 400;
phi = pi/4;

fs = 50*f;
dur = 5/f;

%% generate signal
% 5 cycles with 50 samples per cycle
tt = 0 : 1/fs : dur;
xx = A*cos(2*pi*f*tt + phi);

%% plot
plot(tt,xx)
xlabel( 'Time (s)' )    % labels for x and y axis
ylabel( 'Amplitude' )
title( 'x(t) = A*cos(2\pi f t + \phi)' )
```

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Some Tips for MATLAB programming

- Comment your code (comments start with `%.`).
- Use descriptive names for variables (e.g., `phi`).
- Avoid loops!
- If the above MATLAB code is stored in a file, say `plot_sinusoid.m`, then it can be executed by typing `plot_sinusoid`.
  - Filename must end in `.m`
  - File must be in your working directory,
  - or more generally in MATLAB’s search path.
    - Type `help path` to learn about setting MATLAB’s search path.
Reducing the Sampling Rate

▶ What happens if we reduce the sampling rate? E.g., by setting \( f_s = 5f \);

▶ The sampling rate is not high enough to create a smooth plot. Use at least 20 samples per cycle to get “good-looking” plots!
Very Low Sampling Rates

▶ The plot on the previous plot does not “look” nice, but it still captures the essence of the sinusoidal signal.
▶ A much more serious problem arises when the sampling rate is chosen smaller than twice the frequency of the sinusoid, $f_s < 2f$.
▶ **Example**: assume we try to plot a sinusoidal signal with $f_s = \frac{21}{20} f$.
  ▶ With $f = 400$, the sampling rate is $f_s = 420$.
  ▶ At this rate, we’re collecting just over one sample per cycle.
The resulting plot shows a sinusoid of frequency $f = 20\text{Hz}$ and phase $\phi = -\pi/4$!

This is called **aliasing** and occurs when $f_s < 2f$. 
Energy and Power

- It is often of interest to measured the “strength” of a signal.
- Energy ($E$) and Power ($P$) are used for this purpose.
- **Continuous Time:** for a signal $x(t)$ that is observed between $t = 0$ and $t = T_0$:
  \[
  E = \int_0^{T_0} x^2(t) \, dt \quad P = \frac{1}{T_0} \int_0^{T_0} x^2(t) \, dt
  \]

- **Discrete Time:** Energy and power can be determined from samples $x[n]$ (by approximating $dt \approx \frac{T_0}{N} = \frac{1}{f_s}$):
  \[
  E = \frac{1}{f_s} \sum_{n=0}^{N} x^2[n] \quad P = \frac{1}{N} \sum_{n=0}^{N} x^2[n]
  \]

Note that $N = f_s T_0$. 
Why Complex Signals?

- Complex exponential signals are closely related to sinusoids.
- They eliminate the need for trigonometry ...
- ... and replace it with simple algebra.
  - Complex algebra is really simple - this is not an oxymoron.
- Complex numbers can be represented as vectors.
  - Used to visualize the relationship between sinusoids.
An (unpleasant) Example - Sum of Sinusoids

A typical problem: Express

\[ x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi/2) \]

in the form \( A \cdot \cos(2\pi ft + \phi) \).

Solution: Use trig identity

\[ \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \]

on second term.

This leads to

\[ x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft)\cos(\pi/2) - 4 \cdot \sin(2\pi ft)\sin(\pi/2) \]
\[ = 3 \cdot \cos(2\pi ft) - 4 \cdot \sin(2\pi ft). \]

Compare to what we want:

\[ x(t) = A \cdot \cos(2\pi ft + \phi) \]
\[ = A \cdot \cos(\phi)\cos(2\pi ft) - A \cdot \sin(\phi)\sin(2\pi ft) \]
More Unpleasantness ...

- We can conclude that $A$ and $\phi$ must satisfy

$$A \cdot \cos(\phi) = 3 \text{ and } A \cdot \sin(\phi) = 4.$$ 

- We can find $A$ from

$$A^2 \cdot \cos^2(\phi) + A^2 \cdot \sin^2(\phi) = A^2$$

$$9 + 16 = 25$$

- Thus, $A = 5$.

- Also,

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{4}{3}.$$

- Hence, $\phi \approx \frac{53}{180} \pi \ (53 \degree)$.

- And, $x(t) = 5 \cos(2\pi ft + \frac{53}{180} \pi)$.

- With complex numbers problems of this type are much easier.
Exercise

 Modify the MATLAB code for `plot_sinusoid.m` to plot two sinusoids as well as the sum of the two sinusoids.
  
  ▶ The first sinusoid has parameters $A = 3$, $f = 10$, and $\phi = 0$; it should be plotted in blue.
  ▶ The second sinusoid has parameters $A = 4$, $f = 10$, and $\phi = \pi / 2$; it should be plotted in red.
  ▶ The sum is to be plotted in black.

 ▶ Determine the amplitude, frequency, and phase of the sum of the two sinusoids.
Introduction

The complex exponential signal is defined as

\[ x(t) = A \exp(j(2\pi ft + \phi)). \]

As with sinusoids, \(A\), \(f\), and \(\phi\) are (real-valued) amplitude, frequency, and phase.

By Euler’s relationship, it is closely related to sinusoidal signals

\[ x(t) = A \cos(2\pi ft + \phi) + jA \sin(2\pi ft + \phi). \]

We will leverage the benefits the complex representation provides over sinusoids:

- Avoid trigonometry,
- Replace with simple algebra,
- Visualization in the complex plane.
Complex Plane

\[ x(t) = 1 \cdot \exp(j(2\pi/8t + \pi/4)) \]
3D Plot of Complex Exponential

\[ x(t) = 1 \cdot \exp(j(2\pi/8t + \pi/4)) \]
Expressing Sinusoids through Complex Exponentials

- There are two ways to write a sinusoidal signal in terms of complex exponentials.

- **Real part:**

  \[ A \cos(2\pi ft + \phi) = \text{Re}\{A \exp(j(2\pi ft + \phi))\} \]

- **Inverse Euler:**

  \[ A \cos(2\pi ft + \phi) = \frac{A}{2} (\exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi))) \]

- Both expressions are useful and will be important throughout the course.
Phasors

- Phasors are **not** directed-energy weapons first seen in the original Star Trek movie.
  - That would be *phasers*!
- Phasors are the **complex amplitudes** of complex exponential signals:

  \[ x(t) = A \exp(j(2\pi ft + \phi)) = Ae^{j\phi} \exp(j2\pi ft). \]

- The phasor of this complex exponential is \( X = Ae^{j\phi} \).
- Thus, phasors capture both amplitude \( A \) and phase \( \phi \).
- We can summarize a complex exponential signal through its phasor and frequency: \((Ae^{j\phi}, f)\).
From Sinusoids to Phasors

- A sinusoid can be written as

\[ A \cos(2\pi ft + \phi) = \frac{A}{2} \left( \exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi)) \right) \]

- This can be rewritten to provide

\[ A \cos(2\pi ft + \phi) = \frac{Ae^{j\phi}}{2} \exp(j2\pi ft) + \frac{Ae^{-j\phi}}{2} \exp(-j2\pi ft) \]

- Thus, a sinusoid is composed of two complex exponentials
  - One with frequency \( f \) and phasor \( \frac{Ae^{j\phi}}{2} \),
    - rotates counter-clockwise in the complex plane;
  - one with frequency \( -f \) and phasor \( \frac{Ae^{-j\phi}}{2} \),
    - rotates clockwise in the complex plane;
  - Note that the two phasors are conjugate complexes of each other.
Exercise

- Write

\[ x(t) = 3 \cos(2\pi 10t - \pi/3) \]

as a sum of two complex exponentials.

- For each of the two complex exponentials, find the frequency and the phasor.
Lecture: The Phasor Addition Rule
Problem Statment

- It is often required to add two or more sinusoidal signals.
- When all sinusoids have the same frequency then the problem simplifies.
  - This problem comes up very often, e.g., in AC circuit analysis (ECE 280) and later in the class (chapter 5).
- Starting point: sum of sinusoids

\[ x(t) = A_1 \cos(2\pi ft + \phi_1) + \ldots + A_N \cos(2\pi ft + \phi_N) \]

- Note that all frequencies \( f \) are the same (no subscript).
- Amplitudes \( A_i \) phases \( \phi_i \) are different in general.
- Short-hand notation using summation symbol (\( \sum \)):

\[ x(t) = \sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) \]
The Phasor Addition Rule

- The phasor addition rule implies that there exist an amplitude $A$ and a phase $\phi$ such that

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) = A \cos(2\pi ft + \phi)$$

- **Interpretation**: The sum of sinusoids of the same frequency but different amplitudes and phases is a single sinusoid of the same frequency.
- The phasor addition rule specifies how the amplitude $A$ and the phase $\phi$ depends on the original amplitudes $A_i$ and $\phi_i$.
- **Example**: We showed earlier (by means of an unpleasant computation involving trig identities) that:

$$x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi / 2) = 5 \cos(2\pi ft + 53^\circ)$$
Prerequisites

We will need two simple prerequisites before we can derive the phasor addition rule.

1. Any sinusoid can be written in terms of complex exponentials as follows

   \[ A \cos(2\pi ft + \phi) = \text{Re}\{Ae^{i(2\pi ft + \phi)}\} = \text{Re}\{Ae^{i\phi}e^{i2\pi ft}\}. \]

   Recall that \( Ae^{i\phi} \) is called a phasor (complex amplitude).

2. For any complex numbers \( X_1, X_2, \ldots, X_N \), the real part of the sum equals the sum of the real parts.

   \[ \text{Re}\left\{ \sum_{i=1}^{N} X_i \right\} = \sum_{i=1}^{N} \text{Re}\{X_i\}. \]

   This should be obvious from the way addition is defined for complex numbers.

   \[ (x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2). \]
Deriving the Phasor Addition Rule

**Objective:** We seek to establish that

$$\sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) = A \cos(2\pi ft + \phi)$$

and determine how $A$ and $\phi$ are computed from the $A_i$ and $\phi_i$. 
Deriving the Phasor Addition Rule

► **Step 1**: Using the first pre-requisite, we replace the sinusoids with complex exponentials

\[
\sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) = \sum_{i=1}^{N} \Re\{ A_i e^{j(2\pi ft + \phi_i)} \} = \sum_{i=1}^{N} \Re\{ A_i e^{j\phi_i} e^{j2\pi ft} \}.
\]
**Step 2:** The second prerequisite states that the sum of the real parts equals the real part of the sum

\[
\sum_{i=1}^{N} \text{Re}\{A_i e^{j\phi_i} e^{j2\pi ft}\} = \text{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\}.
\]
Deriving the Phasor Addition Rule

- **Step 3:** The exponential $e^{j2\pi ft}$ appears in all the terms of the sum and can be factored out

$$\text{Re} \left\{ \sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft} \right\} = \text{Re} \left\{ \left( \sum_{i=1}^{N} A_i e^{j\phi_i} \right) e^{j2\pi ft} \right\}$$

- The term $\sum_{i=1}^{N} A_i e^{j\phi_i}$ is just the sum of complex numbers in polar form.
- The sum of complex numbers is just a complex number $X$ which can be expressed in polar form as $X = Ae^{j\phi}$.
- Hence, amplitude $A$ and phase $\phi$ must satisfy

$$Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$$
Deriving the Phasor Addition Rule

- **Note**
  - computing $\sum_{i=1}^{N} A_i e^{j\phi_i}$ requires converting $A_i e^{j\phi_i}$ to rectangular form,
  - the result will be in rectangular form and must be converted to polar form $A e^{j\phi}$. 

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Deriving the Phasor Addition Rule

► **Step 4:** Using $A e^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$ in our expression for the sum of sinusoids yields:

$$\text{Re} \left\{ \left( \sum_{i=1}^{N} A_i e^{j\phi_i} \right) e^{j2\pi ft} \right\} = \text{Re} \left\{ A e^{j\phi} e^{j2\pi ft} \right\} = \text{Re} \left\{ A e^{j(2\pi ft + \phi)} \right\} = A \cos(2\pi ft + \phi).$$

► Note: the above result shows that the sum of sinusoids of the same frequency is a sinusoid of the same frequency.
Applying the Phasor Addition Rule

- Applicable only when sinusoids of same frequency need to be added!

- **Problem:** Simplify

\[ x(t) = A_1 \cos(2\pi ft + \phi_1) + \ldots A_N \cos(2\pi ft + \phi_N) \]

- **Solution:** proceeds in 4 steps

  1. Extract phasors: \( X_i = A_i e^{j\phi_i} \) for \( i = 1, \ldots, N \).
  2. Convert phasors to rectangular form:
     \[ X_i = A_i \cos \phi_i + jA_i \sin \phi_i \] for \( i = 1, \ldots, N \).
  3. Compute the sum: \( X = \sum_{i=1}^{N} X_i \) by adding real parts and imaginary parts, respectively.
  4. Convert result \( X \) to polar form: \( X = Ae^{j\phi} \).

- **Conclusion:** With amplitude \( A \) and phase \( \phi \) determined in the final step

\[ x(t) = A \cos(2\pi ft + \phi) . \]
Example

Problem: Simplify

\[ x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi / 2) \]

Solution:

1. Extract Phasors: \( X_1 = 3e^{j0} = 3 \) and \( X_2 = 4e^{j\pi/2} \).
2. Convert to rectangular form: \( X_1 = 3 \) \( X_2 = 4j \).
3. Sum: \( X = X_1 + X_2 = 3 + 4j \).
4. Convert to polar form: \( A = \sqrt{3^2 + 4^2} = 5 \) and \( \phi = \arctan\left(\frac{4}{3}\right) \approx 53^\circ (\frac{53}{180}\pi) \).

Result:

\[ x(t) = 5 \cos(2\pi ft + 53^\circ) \].
Exercise

Simplify

\[ x(t) = 10 \cos(20\pi t + \frac{\pi}{4}) + 10 \cos(20\pi t + \frac{3\pi}{4}) + 20 \cos(20\pi t - \frac{3\pi}{4}). \]

Answer:

\[ x(t) = 10\sqrt{2} \cos(20\pi t + \pi). \]
Part III

Spectrum Representation of Signals
Lecture: Sums of Sinusoids (of different frequency)
Introduction

➢ To this point we have focused on sinusoids of identical frequency $f$

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i).$$

➢ Note that the frequency $f$ does not have a subscript $i$!

➢ Showed (via phasor addition rule) that the above sum can always be written as a single sinusoid of frequency $f$. 
We will consider sums of sinusoids of different frequencies:

\[ x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

- Note the subscript on the frequencies \( f_i \)!
- This apparently minor difference has significant consequences.
Sum of Two Sinusoids

\[ x(t) = \frac{4}{\pi} \cos(2\pi ft - \pi/2) + \frac{4}{3\pi} \cos(2\pi 3ft - \pi/2) \]
Sum of 25 Sinusoids

\[ x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
MATLAB: Sum of 25 Sinusoids

```matlab
f = 50;
fs = 200*f;

%% generate signals
% 5 cycles with 50 samples per cycle
tt = 0 : 1/fs : 3/f;

xx = zeros(size(tt));
for kk = 1:25
    xx = xx + 4/((2*kk-1)*pi)*cos(2*pi*(2*kk-1)*f*tt - pi/2);
end
```
MATLAB: Sum of 25 Sinusoids

- The `for` loop can be replaced by:

```matlab
kk = (1:25);
xx = 4./((2*kk-1)*pi) * cos(2*pi*(2*kk'-1)*f*tt - pi/2);
```
Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusoids in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids of frequencies $(2n - 1) \cdot f$, and
  - amplitudes $\frac{4}{(2n-1)\pi}$.
- (This is not obvious → Fourier Series).
Non-sinusoidal Signals as Sums of Sinusoids

- The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.
- **Basic idea:** list the “ingredients” of a signal by specifying
  - amplitudes and phases (more specifically, phasors), as well as
  - frequencies of the sinusoids in the sum.
The Spectrum of a Sum of Sinusoids

▶ Begin with the sum of sinusoids introduced earlier

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

where we have broken out a possible constant term.

▶ The term \( A_0 \) can be thought of as corresponding to a sinusoid of frequency zero.

▶ Using the *inverse Euler formula*, we can replace the sinusoids by complex exponentials

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i} \).
The Spectrum of a Sum of Sinusoids (cont’d)

Starting with

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i} \).

The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

\[ X(f) = \{(X_0, 0), \left(\frac{X_1}{2}, f_1\right), \left(\frac{X_1^*}{2}, -f_1\right), \ldots, \left(\frac{X_N}{2}, f_N\right), \left(\frac{X_N^*}{2}, -f_N\right)\}. \]
Example

- Consider the signal

\[ x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4). \]

- Using the inverse Euler relationship

\[
\begin{align*}
x(t) &= 3 + \frac{5}{2} e^{-j\pi/2} \exp(j2\pi 10t) + \frac{5}{2} e^{j\pi/2} \exp(-j2\pi 10t) \\
&\quad + \frac{7}{2} e^{j\pi/4} \exp(j2\pi 25t) + \frac{7}{2} e^{-j\pi/4} \exp(-j2\pi 25t)
\end{align*}
\]

- Hence,

\[ X(f) = \left\{ (3, 0), \left(\frac{5}{2} e^{-j\pi/2}, 10\right), \left(\frac{5}{2} e^{j\pi/2}, -10\right), \left(\frac{7}{2} e^{j\pi/4}, 25\right), \left(\frac{7}{2} e^{-j\pi/4}, -25\right) \right\} \]
Exercise

Find the spectrum of the signal:

\[ x(t) = 6 + 4\cos(10\pi t + \pi/3) + 5\cos(20\pi t - \pi/7). \]
Lecture: From Time-Domain to Frequency-Domain and back
Time-domain and Frequency-domain

- Signals are *naturally* observed in the time-domain.
- A signal can be illustrated in the time-domain by plotting it as a function of time.
- The frequency-domain provides an alternative perspective of the signal based on sinusoids:
  - Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
  - The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
  - Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of these parameters.
    - Actually, we list pairs of complex amplitudes \((Ae^{j\phi})\) and frequencies \(f\) and refer to this list as \(X(f)\).
Time-domain and Frequency-domain

- It is possible (but not necessarily easy) to find $X(f)$ from $x(t)$: this is called Fourier or spectrum analysis.
- Similarly, one can construct $x(t)$ from the spectrum $X(f)$: this is called Fourier synthesis.
- Notation: $x(t) \leftrightarrow X(f)$.
- Example (from last time):
  - **Time-domain**: signal

\[
x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4).
\]

  - **Frequency Domain**: spectrum

\[
X(f) = \{(3, 0), (\frac{5}{2}e^{-j\pi/2}, 10), (\frac{5}{2}e^{j\pi/2}, -10), (\frac{7}{2}e^{j\pi/4}, 25), (\frac{7}{2}e^{-j\pi/4}, -25)\}
\]
Plotting a Spectrum

▶ To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.
▶ Sometimes the phase is plotted versus frequency as well.
Why Bother with the Frequency-Domain?

- In many applications, the frequency contents of a signal is very important.
  - For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
  - Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.
- Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).
- We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.
  - Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.
Example: Original signal
Example: Corrupted signal
Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- **Ingredients** are given by the spectrum

\[ X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\} \]

Each pair indicates one complex exponential component by listing its frequency and complex amplitude.

- **Instructions** for combining the ingredients and producing the (time-domain) signal:

\[ x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t). \]

- Always simplify the expression you obtain!
Example

- Problem: Find the signal $x(t)$ corresponding to

$$X(f) = \{(3, 0), \left(\frac{5}{2}e^{-j\pi/2}, 10\right), \left(\frac{5}{2}e^{j\pi/2}, -10\right), \left(\frac{7}{2}e^{j\pi/4}, 25\right), \left(\frac{7}{2}e^{-j\pi/4}, -25\right)\}$$

- Solution:

$$x(t) = 3 + \frac{5}{2}e^{-j\pi/2}e^{2\pi 10t} + \frac{5}{2}e^{j\pi/2}e^{-j2\pi 10t} + \frac{7}{2}e^{j\pi/4}e^{2\pi 25t} + \frac{7}{2}e^{-j\pi/4}e^{-j2\pi 25t}$$

- Which simplifies to:

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$
Exercise

Find the signal that has the spectrum:

\[ X(f) = \{(5, 0), (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \left(\frac{5}{2}e^{j\pi/4}, 15\right), \left(\frac{5}{2}e^{-j\pi/4}, -15\right)\} \]
Analysis: From Time to Frequency-Domain

The objective of spectrum or Fourier analysis is to find the spectrum of a time-domain signal.

We will restrict ourselves to signals $x(t)$ that are sums of sinusoids

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

We have already shown that such signals have spectrum:

$$X(f) = \{(X_0, 0), \left(\frac{1}{2} X_1, f_1\right), \left(\frac{1}{2} X_1^*, -f_1\right), \ldots, \left(\frac{1}{2} X_N, f_N\right), \left(\frac{1}{2} X_N^*, -f_N\right)\}$$

where $X_0 = A_0$ and $X_i = A_i e^{i\phi_i}$.

We will investigate some interesting signals that can be written as a sum of sinusoids.
Beat Notes

► Consider the signal

\[ x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t). \]

► This signal does not have the form of a sum of sinusoids; hence, we can not determine it’s spectrum immediately.
MATLAB Code for Beat Notes

% Parameters
fs = 8192;
dur = 2;

f1 = 5;
f2 = 400;
A = 2;

NP = round(2*fs/f1);  % number of samples to plot

% time axis and signal
tt=0:1/fs:dur;
xx = A*cos(2*pi*f1*tt).*cos(2*pi*f2*tt);

plot(tt(1:NP),xx(1:NP),tt(1:NP),A*cos(2*pi*f1*tt(1:NP)),'r')
xlabel('Time (s)')
ylabel('Amplitude')
grid
Beat Notes as a Sum of Sinusoids

▶ Using the inverse Euler relationships, we can write

\[
x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t)
\]

\[
= 2 \cdot \frac{1}{2} \cdot (e^{j2\pi 5t} + e^{-j2\pi 5t}) \cdot \frac{1}{2} \cdot (e^{j2\pi 400t} + e^{-j2\pi 400t}).
\]

▶ Multiplying out yields:

\[
x(t) = \frac{1}{2} (e^{j2\pi 405t} + e^{-j2\pi 405t}) + \frac{1}{2} (e^{j2\pi 395t} + e^{-j2\pi 395t}).
\]

▶ Applying Euler’s relationship, lets us write:

\[
x(t) = \cos(2\pi 405t) + \cos(2\pi 395t).
\]
Spectrum of Beat Notes

We were able to rewrite the beat notes as a sum of sinusoids

\[ x(t) = \cos(2\pi 405t) + \cos(2\pi 395t). \]

Note that the frequencies in the sum, 395 Hz and 405 Hz, are the sum and difference of the frequencies in the original product, 5 Hz and 400 Hz.

It is now straightforward to determine the spectrum of the beat notes signal:

\[ X(f) = \left\{ \left( \frac{1}{2}, 405 \right), \left( \frac{1}{2}, -405 \right), \left( \frac{1}{2}, 395 \right), \left( \frac{1}{2}, -395 \right) \right\} \]
Spectrum of Beat Notes
Amplitude Modulation

- Amplitude Modulation (AM) is used in communication systems.
- The objective of amplitude modulation is to move the spectrum of a signal $m(t)$ from low frequencies to high frequencies.
  - The message signal $m(t)$ may be a piece of music; its spectrum occupies frequencies below 20 KHz.
  - For transmission by an AM radio station this spectrum must be moved to approximately 1 MHz.
Amplitude Modulation

- Conventional amplitude modulation proceeds in two steps:
  1. A constant $A$ is added to $m(t)$ such that $A + m(t) > 0$ for all $t$.
  2. The sum signal $A + m(t)$ is multiplied by a sinusoid $\cos(2\pi f_c t)$, where $f_c$ is the radio frequency assigned to the station.

- Consequently, the transmitted signal has the form:

$$x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$
Amplitude Modulation

- We are interested in the spectrum of the AM signal.
- However, we cannot (yet) compute $X(f)$ for arbitrary message signals $m(t)$.
- For the special case $m(t) = \cos(2\pi f_m t)$ we can find the spectrum.
  - To mimic the radio case, $f_m$ would be a frequency in the audible range.
- As before, we will first need to express the AM signal $x(t)$ as a sum of sinusoids.
Amplitude Modulated Signal

For $m(t) = \cos(2\pi f_m t)$, the AM signal equals

$$x(t) = (A + \cos(2\pi f_m t)) \cdot \cos(2\pi f_c t).$$

This simplifies to

$$x(t) = A \cdot \cos(2\pi f_c t) + \cos(2\pi f_m t) \cdot \cos(2\pi f_c t).$$

Note that the second term of the sum is a beat notes signal with frequencies $f_m$ and $f_c$.

We know that beat notes can be written as a sum of sinusoids with frequencies equal to the sum and difference of $f_m$ and $f_c$:

$$x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2} \cos(2\pi (f_c + f_m) t) + \frac{1}{2} \cos(2\pi (f_c - f_m) t).$$
Plot of Amplitude Modulated Signal

For $A = 2$, $f_m = 50$, and $f_c = 400$, the AM signal is plotted below.
The AM signal is given by

\[ x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2} \cos(2\pi (f_c + f_m) t) + \frac{1}{2} \cos(2\pi (f_c - f_m) t) \]

Thus, its spectrum is

\[ X(f) = \{ (\frac{A}{2}, f_c), (\frac{A}{2}, -f_c), (\frac{1}{4}, f_c + f_m), (\frac{1}{4}, -f_c - f_m), (\frac{1}{4}, f_c - f_m), (\frac{1}{4}, -f_c + f_m) \} \]
Spectrum of Amplitude Modulated Signal

For $A = 2$, $fm = 50$, and $fc = 400$, the spectrum of the AM signal is plotted below.
Spectrum of Amplitude Modulated Signal

- It is interesting to compare the spectrum of the signal before modulation and after multiplication with \(\cos(2\pi f_c t)\).
- The signal \(s(t) = A + m(t)\) has spectrum
  \[
  S(f) = \{(A, 0), \left(\frac{1}{2}, 50\right), \left(\frac{1}{2}, -50\right)\}.
  \]
- The modulated signal \(x(t)\) has spectrum
  \[
  X(f) = \{(\frac{A}{2}, 400), (\frac{A}{2}, -400),
  (\frac{1}{4}, 450), (\frac{1}{4}, -450), (\frac{1}{4}, 350), (\frac{1}{4}, -350)\}.
  \]
- Both are plotted on the next page.
Spectrum before and after AM
Comparison of the two spectra shows that amplitude module indeed moves a spectrum from low frequencies to high frequencies.

Note that the shape of the spectrum is precisely preserved.

Amplitude modulation can be described concisely by stating:

- Half of the original spectrum is shifted by $f_c$ to the right, and the other half is shifted by $f_c$ to the left.

**Question:** How can you get the original signal back so that you can listen to it.

- This is called demodulation.
Lecture: Periodic Signals
What are Periodic Signals?

- A signal $x(t)$ is called periodic if there is a constant $T_0$ such that
  $$x(t) = x(t + T_0) \text{ for all } t.$$  

- In other words, a periodic signal repeats itself every $T_0$ seconds.

- The interval $T_0$ is called the fundamental period of the signal.

- The inverse of $T_0$ is the fundamental frequency of the signal.

- Example:
  - A sinusoidal signal of frequency $f$ is periodic with period
    $$T_0 = 1/f.$$
Harmonic Frequencies

- Consider a sum of sinusoids:

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

- A special case arises when we constrain all frequencies \( f_i \) to be integer multiples of some frequency \( f_0 \):

\[ f_i = i \cdot f_0. \]

- The frequencies \( f_i \) are then called **harmonic** frequencies of \( f_0 \).

- We will show that sums of sinusoids with frequencies that are harmonics are periodic.
Harmonic Signals are Periodic

- To establish periodicity, we must show that there is $T_0$ such that $x(t) = x(t + T_0)$.

- Begin with

$$x(t + T_0) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i (t + T_0) + \phi_i)$$

- Now, let $f_0 = 1 / T_0$ and use the fact that frequencies are harmonics: $f_i = i \cdot f_0$. 
Harmonic Signals are Periodic

- Then, \( f_i \cdot T_0 = i \cdot f_0 \cdot T_0 = i \) and hence

\[
x(t + T_0) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i)
\]

\[
= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi i + \phi_i)
\]

- We can drop the \( 2\pi i \) terms and conclude that

\[x(t + T_0) = x(t).\]

- **Conclusion:** A signal of the form

\[
x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i \cdot f_0 t + \phi_i)
\]

is periodic with period \( T_0 = 1 / f_0 \).
Finding the Fundamental Frequency

- Often one is given a set of frequencies $f_1, f_2, \ldots, f_N$ and is required to find the fundamental frequency $f_0$.
- Specifically, this means one must find a frequency $f_0$ and integers $n_1, n_2, \ldots, n_N$ such that all of the following equations are met:

\[
\begin{align*}
    f_1 &= n_1 \cdot f_0 \\
    f_2 &= n_2 \cdot f_0 \\
    &\vdots \\
    f_N &= n_N \cdot f_0
\end{align*}
\]

- Note that there isn’t always a solution to the above problem.
  - However, if all frequencies are integers a solution exists.
  - Even if all frequencies are rational a solution exists.
Example

- Find the fundamental frequency for the set of frequencies $f_1 = 12$, $f_2 = 27$, $f_3 = 51$.
- Set up the equations:

\[
12 = n_1 \cdot f_0 \\
27 = n_2 \cdot f_0 \\
51 = n_3 \cdot f_0
\]

- Try the solution $n_1 = 1$; this would imply $f_0 = 12$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 2$; this would imply $f_0 = 6$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 3$; this would imply $f_0 = 4$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 4$; this would imply $f_0 = 3$. This can satisfy the other two equations with $n_2 = 9$ and $n_3 = 17$. 
Example

- Note that the three sinusoids complete a cycle at the same time at $T_0 = 1/f_0 = 1/3\text{s}$.

![Graph showing sinusoidal waves with different frequencies and their sum]
A Few Things to Note

- Note that the fundamental frequency $f_0$ that we determined is the greatest common divisor (gcd) of the original frequencies.
  - $f_0 = 3$ is the gcd of $f_1 = 12$, $f_2 = 27$, and $f_3 = 51$.
- The integers $n_i$ are the number of full periods (cycles) the sinusoid of frequency $f_i$ completes in the fundamental period $T_0 = 1/f_0$.
  - For example, $n_1 = f_1 \cdot T_0 = f_1 \cdot 1/f_0 = 4$.
  - The sinusoid of frequency $f_1$ completes $n_1 = 4$ cycles during the period $T_0$. 
Exercise

- Find the fundamental frequency for the set of frequencies $f_1 = 2, f_2 = 3.5, f_3 = 5$. 

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Fourier Series

- We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.
- One can turn this statement around and arrive at a very important result:

  \[
  \text{Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.}
  \]

- The resulting sum is called the Fourier Series of the signal.
- Put differently, a periodic signal can always be written in the form

  \[
  x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_0 t + \phi_i) \\
  = X_0 + \sum_{i=1}^{N} X_i e^{j2\pi f_0 t} + X_i^* e^{-j2\pi f_0 t}
  \]

  with \( X_0 = A_0 \) and \( X_i = \frac{A_i}{2} e^{j\phi_i} \).
Fourier Series

For a periodic signal the complex amplitudes $X_i$ can be computed using a (relatively) simple formula.

Specifically, for a periodic signal $x(t)$ with fundamental period $T_0$ the complex amplitudes $X_i$ are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi it/T_0} dt.$$ 

Note that the integral above can be evaluated over any interval of length $T_0$. 

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Example: Square Wave

A square wave signal is periodic and between $t = 0$ and $t = T_0$ it equals

$$x(t) = \begin{cases} 
1 & 0 \leq t < \frac{T_0}{2} \\
-1 & \frac{T_0}{2} \leq t < T_0
\end{cases}$$

From the Fourier Series expansion it follows that $x(t)$ can be written as

$$x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$
25-Term Approximation to Square Wave

\[ x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
Lecture: Operations on the Spectrum
Operations on the Spectrum

- Signals are usually manipulated (processed) in the time domain.
  - E.g., adding two signals, taking the derivative of a signal.
- We will investigate how common operations on signals in the time-domain affect the spectrum of the signal.
  - Recall the notation we use to indicate that signal $x(t)$ and spectrum $X(f)$ are related:

$$x(t) \leftrightarrow X(f).$$
Multiplication by a Constant

- Let \( x(t) \leftrightarrow X(f) \).
  - \( x(t) = \sum_n X_n e^{j2\pi f_n t} \) and, thus, \( X(f) = \{(X_n, f_n)\}_n \)
- We form a new signal \( y(t) \) by multiplying \( x(t) \) by a constant \( c \):
  \[
y(t) = c \cdot x(t) = \sum_n c \cdot X_n e^{j2\pi f_n t}.
  \]
- Then,
  \[
  Y(f) = c \cdot X(f) = \{(c \cdot X_n, f_n)\}_n.
  \]
Addition of Two Signals

- Let \( x_1(t) \leftrightarrow X_1(f) \) and \( x_2(t) \leftrightarrow X_2(f) \).
- Form \( y(t) = x_1(t) + x_2(t) \).
- Then,

\[
Y(f) = X_1(f) + X_2(f)
\]

- To compute this addition,
  - Any spectral component that appears only in \( X_1(f) \) or only in \( X_2(f) \) is copied to the output spectrum \( Y(f) \).
  - Components that appear in both input spectra must be added (phasor addition).

\[
x_1(t) + x_2(t) \leftrightarrow X_1(f) + X_2(f)
\]
Example

Let

\[ x_1(t) = 3 + \sqrt{2} \cos(2\pi 3t + \pi/4) \]

\[ \uparrow \]

\[ X_1(f) = \{(3, 0), \left(\frac{\sqrt{2}}{2} e^{j\pi/4}, 3\right), \left(\frac{\sqrt{2}}{2} e^{-j\pi/4}, -3\right)\} \]

\[ x_2(t) = \sqrt{2} \cos(2\pi 3t - \pi/4) + 4 \cos(2\pi 4t) \]

\[ \uparrow \]

\[ X_2(f) = \left\{ \left(\frac{\sqrt{2}}{2} e^{-j\pi/4}, 3\right), \left(\frac{\sqrt{2}}{2} e^{j\pi/4}, -3\right), (2, 4), (2, -4) \right\} \]

Then

\[ Y(f) = \{(3, 0), (1, 3), (1, -3), (2, 4), (2, -4)\} \]
Time Delay

- Let \( x(t) \sum_n X_n e^{j2\pi f_n t} \).
- Delay \( x(t) \) by \( \tau \) to form \( y(t) = x(t - \tau) \).
- Then,

\[
y(t) = \sum_n X_n e^{j2\pi f_n (t-\tau)} \\
= \sum_n X_n e^{j(2\pi f_n t - 2\pi f_n \tau)} \\
= \sum_n X_n e^{-j2\pi f_n \tau} e^{j2\pi f_n t}
\]

\[
x(t - \tau) \leftrightarrow e^{j2\pi f \tau} \cdot X(f) = \{(e^{j2\pi f \tau}) \cdot X_n, f_n\}_n
\]
Frequency Shifting

- Let \( x(t) \sum_n X_n e^{j2\pi f_n t} \).
- Multiply \( x(t) \) by \( e^{j2\pi f_c t} \) to form \( y(t) = x(t) \cdot e^{j2\pi f_c t} \).
- Then,

\[
y(t) = \sum_n X_n e^{j2\pi f_n t} \cdot e^{j2\pi f_c t} \\
= \sum_n X_n e^{j2\pi (f_n + f_c) t}
\]

\[x(t) \cdot e^{j2\pi f_c t} \leftrightarrow X(f + f_c) = \{(X_n, f_n + f_c)\}^n\]
Differentiating a Signal

Let \( x(t) \sum_n X_n e^{j2\pi f_n t} \).

Form \( y(t) = \frac{dx(t)}{dt} \).

Then,

\[
y(t) = \frac{d}{dt} \sum_n X_n e^{j2\pi f_n t}
\]

\[
= \sum_n X_n \frac{d}{dt} e^{j2\pi f_n t}
\]

\[
= \sum_n X_n (j2\pi f_n) e^{j2\pi f_n t}
\]

\[
\frac{dx(t)}{dt} \leftrightarrow (j2\pi f) \cdot X(f) = \{(j2\pi f_n X_n, f_n)\}_n
\]
Exercise

- Let $x(t) = 5 + 4 \cos(2\pi 10t) + 6 \cos(2\pi 20t)$.
- Compute the spectrum $X(f)$ of $x(t)$.
- Find the spectrum of the signal $y(t) = x(t) + x(t - 0.05)$.
- Find the spectrum of the signal $z(t) = y(t) \cdot \cos(2\pi 20t)$. 
Limitations of Sum-of-Sinusoid Signals

- So far, we have considered only signals that can be written as a sum of sinusoids.

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

- For such signals, we are able to compute the spectrum.
- Note, that signals of this form
  - are assumed to last forever, i.e., for \(-\infty < t < \infty\),
  - and their spectrum never changes.
- While such signals are important and useful conceptually, they don’t describe real-world signals well.
- Real-world signals
  - are of finite duration,
  - their spectrum changes over time.
Musical Notation

- Musical notation ("sheet music") provides a way to represent real-world signals: a piece of music.
- As you know, sheet music
  - places notes on a scale to reflect the frequency of the tone to be played,
  - uses differently shaped note symbols to indicate the duration of each tone,
  - provides the order in which notes are to be played.
- In summary, musical notation captures how the spectrum of the music-signal changes over time.
- We cannot write signals whose spectrum changes with time as a sum of sinusoids.
  - A static spectrum is insufficient to describe such signals.
- Alternative: time-frequency spectrum
Example: Musical Scale

<table>
<thead>
<tr>
<th>Note</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>262</td>
<td>294</td>
<td>330</td>
<td>349</td>
<td>392</td>
<td>440</td>
<td>494</td>
<td>523</td>
</tr>
</tbody>
</table>

Table: Musical Notes and their Frequencies
Example: Musical Scale

- If we play each of the notes for 250 ms, then the resulting signal can be summarized in the time-frequency spectrum below.
MATLAB Spectrogram Function

- MATLAB has a function `spectrogram` that can be used to compute the time-frequency spectrum for a given signal.
  - The resulting plots are similar to the one for the musical scale on the previous slide.

- Typically, you invoke this function as
  ```matlab
  spectrogram( xx, 256, 128, 256, fs, 'yaxis'),
  ```
  where `xx` is the signal to be analyzed and `fs` is the sampling frequency.

- The spectrogram for the musical scale is shown on the next slide.
Spectrogram: Musical Scale

- The color indicates the magnitude of the spectrum at a given time and frequency.
Chirp Signals

- **Objective:** construct a signal such that its frequency increases with time.

- **Starting Point:** A sinusoidal signal has the form:

  \[ x(t) = A \cos(2\pi f_0 t + \phi). \]

- We can consider the argument of the cos as a time-varying phase function

  \[ \Psi(t) = 2\pi f_0 t + \phi. \]

- **Question:** What happens when we allow more general functions for \( \Psi(t) \)?
  - For example, let

    \[ \Psi(t) = 700\pi t^2 + 440\pi t + \phi. \]
**Spectrogram: \( \cos(\Psi(t)) \)**

- **Question:** How is the time-frequency spectrum related to \( \Psi(t) \)?
Instantaneous Frequency

- For a regular sinusoid, $\Psi(t) = 2\pi f_0 t + \phi$ and the frequency equals $f_0$.
- This suggests as a possible relationship between $\Psi(t)$ and $f_0$

$$f_0 = \frac{1}{2\pi} \frac{d}{dt} \Psi(t).$$

- If the above derivative is not a constant, it is called the instantaneous frequency of the signal, $f_i(t)$.
- **Example:** For $\Psi(t) = 700\pi t^2 + 440\pi t + \phi$ we find

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} (700\pi t^2 + 440\pi t + \phi) = 700t + 220.$$

- This describes precisely the red line in the spectrogram on the previous slide.
Constructing a Linear Chirp

- **Objective:** Construct a signal such that its frequency is initially $f_1$ and increases linear to $f_2$ after $T$ seconds.

- **Solution:** The above suggests that

  \[ f_i(t) = \frac{f_2 - f_1}{T} t + f_1. \]

- Consequently, the phase function $\Psi(t)$ must be

  \[ \Psi(t) = 2\pi \frac{f_2 - f_1}{2T} t^2 + 2\pi f_1 t + \phi \]

- Note that $\phi$ has no influence on the spectrum; it is usually set to 0.
**Example:** Construct a linear chirp such that the frequency decreases from 1000 Hz to 200 Hz in 2 seconds.

The desired signal must be

\[ x(t) = \cos(-2\pi 200t^2 + 2\pi 1000t). \]
Exercise

- Construct a linear chirp such that the frequency increases from 50 Hz to 200 Hz in 3 seconds.
- Sketch the time-frequency spectrum of the following signal

\[ x(t) = \cos(2\pi 800t + 100 \cos(2\pi 4t)) \]
Part IV

Sampling of Signals
Lecture: Introduction to Sampling
Introduction to Sampling

Sampling and Discrete-Time Signals

- MATLAB, and other digital processing systems, can not process continuous-time signals.
- Instead, MATLAB requires the continuous-time signal to be converted into a discrete-time signal.
- The conversion process is called sampling.
- To sample a continuous-time signal, we evaluate it at a discrete set of times $t_n = nT_s$, where
  - $n$ is an integer,
  - $T_s$ is called the sampling period (time between samples),
  - $f_s = 1 / T_s$ is the sampling rate (samples per second).
Sampling and Discrete-Time Signals

- Sampling results in a sequence of samples

\[ x(nT_s) = A \cdot \cos(2\pi fnT_s + \phi). \]

- Note that the independent variable is now \( n \), not \( t \).
- To emphasize that this is a discrete-time signal, we write

\[ x[n] = A \cdot \cos(2\pi fnT_s + \phi). \]

- Sampling is a straightforward operation.
- We will see that the sampling rate \( f_s \) must be chosen with care!
Sampled Signals in MATLAB

- Note that we have worked with sampled signals whenever we have used MATLAB.

- For example, we use the following MATLAB fragment to generate a sinusoidal signal:

```matlab
fs = 100;
tt = 0:1/fs:3;
xx = 5*cos(2*pi*2*tt + pi/4);
```

- The resulting signal $xx$ is a discrete-time signal:
  - The vector $xx$ contains the samples, and
  - the vector $tt$ specifies the sampling instances: $0, 1/f_s, 2/f_s, \ldots, 3$.

- We will now turn our attention to the impact of the sampling rate $f_s$. 
Example: Three Sinuoids

» **Objective:** In MATLAB, compute sampled versions of three sinusoids:

1. \( x(t) = \cos(2\pi t + \pi/4) \)
2. \( x(t) = \cos(2\pi 9t - \pi/4) \)
3. \( x(t) = \cos(2\pi 11t + \pi/4) \)

» The sampling rate for all three signals is \( f_s = 10 \).
% plot_SamplingDemo - Sample three sinusoidal signals to demonstrate the impact of sampling

%% set parameters
fs = 10;
dur = 10;

%% generate signals
tt = 0:1/fs:dur;
xx1 = cos(2*pi*tt+pi/4);
xx2 = cos(2*pi*9*tt-pi/4);
xx3 = cos(2*pi*11*tt+pi/4);

%% plot
plot(tt,xx1,:o',tt,xx2,:x',tt,xx3,:+');
xlabel('Time (-s)')
grid
legend('f=1','f=9','f=11','Location','EastOutside')
What happened?

- The samples for all three signals are identical: how is that possible?
- Is there a “bug” in the MATLAB code?
  - No, the code is correct.
- **Suspicion**: The problem is related to our choice of sampling rate.
  - To test this suspicion, repeat the experiment with a different sampling rate.
  - We also reduce the duration to keep the number of samples constant - that keeps the plots reasonable.
MATLAB code

```matlab
% plot_SamplingDemoHigh - Sample three sinusoidal signals to
demonstrate the impact of sampling

%% set parameters
fs = 100;
dur = 1;

%% generate signals
tt = 0:1/fs:dur;
xx1 = cos(2*pi*tt+pi/4);
xx2 = cos(2*pi*9*tt-pi/4);
xx3 = cos(2*pi*11*tt+pi/4);

%% plots
plot(tt,xx1,'-*',tt,xx2,'-x',tt,xx3,'-+',
     tt(1:10:end), xx1(1:10:end),'ok');
grid
xlabel('Time (s)')
legend('f=1','f=9','f=11','f_s=10','Location','EastOutside')
```
Resulting Plot

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The Influence of the Sampling Rate

- Now the three sinusoids are clearly distinguishable and lead to different samples.
- Since the only parameter we changed is the sampling rate $f_s$, it must be responsible for the ambiguity in the first plot.
- Notice also that every 10-th sample (marked with a black circle) is identical for all three sinusoids.
  - Since the sampling rate was 10 times higher for the second plot, this explains the first plot.
- It is useful to investigate the effect of sampling mathematically, to understand better what impact it has.
  - To do so, we focus on sampling sinusoidal signals.
Sampling a Sinusoidal Signal

- A continuous-time sinusoid is given by
  \[ x(t) = A \cos(2\pi ft + \phi). \]

- When this signal is sampled at rate \( f_s \), we obtain the discrete-time signal
  \[ x[n] = A \cos(2\pi fn/f_s + \phi). \]

- It is useful to define the normalized or digital frequency \( \hat{f} = f/f_s \), so that
  \[ x[n] = A \cos(2\pi \hat{f}n + \phi). \]
Three Cases

- We will distinguish between three cases:
  1. $0 \leq \hat{f} \leq 1/2$ (Oversampling, this is what we want!)
  2. $1/2 < \hat{f} \leq 1$ (Undersampling, folding)
  3. $1 < \hat{f} \leq 3/2$ (Undersampling, aliasing)

- This captures the three situations addressed by the first example:
  1. $f = 1, f_s = 10 \Rightarrow \hat{f} = 1/10$
  2. $f = 9, f_s = 10 \Rightarrow \hat{f} = 9/10$
  3. $f = 11, f_s = 10 \Rightarrow \hat{f} = 11/10$

- We will see that all three cases lead to identical samples.
Oversampling

- When the sampling rate is such that $0 \leq \hat{f} \leq 1/2$, then the samples of the sinusoidal signal are given by

$$x[n] = A \cos(2\pi \hat{f} n + \phi).$$

- This cannot be simplified further.
- It provides our base-line.
- Oversampling is the desired behaviour!
Undersampling, Aliasing

▶ When the sampling rate is such that $1 < \hat{f} \leq 3/2$, then we define the **apparent frequency** $\hat{f}_a = \hat{f} - 1$.

▶ Notice that $0 < \hat{f}_a \leq 1/2$ and $\hat{f} = \hat{f}_a + 1$.
  
  ▶ For $f = 11$, $f_s = 10 \Rightarrow \hat{f} = 11/10 \Rightarrow \hat{f}_a = 1/10$.

▶ The samples of the sinusoidal signal are given by

$$x[n] = A \cos(2\pi \hat{f}_a n + \phi) = A \cos(2\pi (1 + \hat{f}_a)n + \phi).$$

▶ Expanding the terms inside the cosine,

$$x[n] = A \cos(2\pi \hat{f}_a n + 2\pi n + \phi) = A \cos(2\pi \hat{f}_a n + \phi)$$

▶ **Interpretation:** The samples are identical to those from a sinusoid with frequency $f = \hat{f}_a \cdot f_s$ and phase $\phi$. 

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Undersampling, Folding

- When the sampling rate is such that $1/2 < \hat{f} \leq 1$, then we introduce the apparent frequency $\hat{f}_a = 1 - \hat{f}$; again $0 < \hat{f}_a \leq 1/2$; also $\hat{f} = 1 - \hat{f}_a$.
  - For $f = 9$, $f_s = 10 \Rightarrow \hat{f} = 9/10 \Rightarrow \hat{f}_a = 1/10$.
- The samples of the sinusoidal signal are given by
  \[ x[n] = A \cos(2\pi\hat{f}n + \phi) = A \cos(2\pi(1 - \hat{f}_a)n + \phi). \]
- Expanding the terms inside the cosine,
  \[ x[n] = A \cos(-2\pi\hat{f}_a n + 2\pi n + \phi) = A \cos(-2\pi\hat{f}_a n + \phi) \]
- Because of the symmetry of the cosine, this equals
  \[ x[n] = A \cos(2\pi\hat{f}_a n - \phi). \]
- **Interpretation:** The samples are identical to those from a sinusoid with frequency $f = \hat{f}_a \cdot f_s$ and phase $-\phi$ (phase reversal).
Introduction to Sampling

Sampling Higher-Frequency Sinusoids

- For sinusoids of even higher frequencies \( f \), either folding or aliasing occurs.
- As before, let \( \hat{f} \) be the normalized frequency \( f / f_s \).
- Decompose \( \hat{f} \) into an integer part \( N \) and fractional part \( f_p \).
  - **Example:** If \( \hat{f} \) is 5.7 then \( N \) equals 5 and \( f_p \) is 0.7.
  - Notice that \( 0 \leq f_p < 1 \), always.
- **Phase Reversal** occurs when the phase of the sampled sinusoid is the negative of the phase of the continuous-time sinusoid.
- We distinguish between
  - **Folding** occurs when \( f_p > 1/2 \). Then the apparent frequency \( \hat{f}_a \) equals \( 1 - f_p \) and phase reversal occurs.
  - **Aliasing** occurs when \( f_p \leq 1/2 \). Then the apparent frequency is \( \hat{f}_a = f_p \); no phase reversal occurs.
Examples

▶ For the three sinusoids considered earlier:
1. \( f = 1, \phi = \pi / 4, f_s = 10 \Rightarrow \hat{f} = 1/10 \)
2. \( f = 9, \phi = -\pi / 4, f_s = 10 \Rightarrow \hat{f} = 9/10 \)
3. \( f = 11, \phi = \pi / 4, f_s = 10 \Rightarrow \hat{f} = 11/10 \)

▶ The first case, represents oversampling: The apparent frequency \( \hat{f}_a = \hat{f} \) and no phase reversal occurs.

▶ The second case, represents folding: The apparent \( \hat{f}_a \) equals \( 1 - \hat{f} \) and phase reversal occurs.

▶ In the final example, the fractional part of \( \hat{f} = 1/10 \). Hence, this case represents alising; no phase reversal occurs.
Exercise

The discrete-time sinusoidal signal

\[ x[n] = 5 \cos(2\pi 0.2n - \frac{\pi}{4}) \]

was obtained by sampling a continuous-time sinusoid of the form

\[ x(t) = A \cos(2\pi ft + \phi) \]

at the sampling rate \( f_s = 8000 \) Hz.

1. Provide three different sets of parameters \( A, f, \) and \( \phi \) for the continuous-time sinusoid that all yield the discrete-time sinusoid above when sampled at the indicated rate. The parameter \( f \) must satisfy \( 0 < f < 12000 \) Hz in all three cases.

2. For each case indicate if the signal is undersampled or oversampled and if aliasing or folding occurred.
Two experiments to illustrate the effects that sampling introduces:

1. Sampling a chirp signal.
2. Sampling a rotating phasor.
Introduction to Sampling

Experiment: Sampling a Chirp Signal

- **Objective:** Directly observe folding and aliasing by means of a chirp signal.

- **Experiment Set-up:**
  - Set sampling rate. Baseline: $f_s = 44.1$KHz (oversampled), Comparison: $f_s = 8.192$KHz (undersampled)
  - Generate a (sampled) chirp signal with instantaneous frequency increasing from 0 to 20KHz in 10 seconds.
  - Evaluate resulting signal by
    - playing it through the speaker,
    - plotting the periodogram.

- **Expected Outcome?**

- **Expected Outcome:**
  - Directly observe folding and aliasing in second part of experiment.
Introduction to Sampling

Periodogram of undersampled Chirp
%% Parameters
fs = 8192;  % 44.1KHz for oversampling, 8192 for undersampling

% chipt: 0 to 20KHz in 10 seconds
fstart = 0;
fend = 20e3;
dur = 10;

%% generate signal
tt = 0:1/fs:dur;
psi = 2*pi*(fend-fstart)/(2*dur)*tt.^2;  % phase function
xx = cos(psi);

%% spectrogram
spectrogram( xx, 256, 128, 256, fs,'yaxis');

%% play sound
soundsc( xx, fs);
Apparent and Normalized Frequency
Experiment: Sampling a Rotating Phasor

- **Objective:** Investigate sampling effects when we can distinguish between positive and negative frequencies.

- **Experiment Set-up:**
  - Animation: rotating phasor in the complex plane.
  - Sampling rate describes the number of “snap-shots” per second (strobes).
  - Frequency the number of times the phasor rotates per second.
    - positive frequency: counter-clockwise rotation.
    - negative frequency: clockwise rotation.

- **Expected Outcome?**
- **Expected Outcome:**
  - Folding: leads to reversal of direction.
  - Aliasing: same direction but apparent frequency is lower than true frequency.
## True and Apparent Frequency

$$f_s = 10$$

<table>
<thead>
<tr>
<th>True Frequency</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>9.5</th>
<th>10</th>
<th>10.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apparent Frequency</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

- Note, that instead of folding we observe negative frequencies.
  - Occurs when true frequency equals 9.5 in above example.
%% parameters
fs = 10;  \hspace{1em} \% sampling rate in frames per second
dur = 10;  \hspace{1em} \% signal duration in seconds

ff = 9.5;  \hspace{1em} \% frequency of rotating phasor
phi = 0;  \hspace{1em} \% initial phase of phasor
A = 1;  \hspace{1em} \% amplitude

%% Prepare for plot
TitleString = sprintf('Rotating Phasor: \textit{f}_d = %5.2f', ff/fs);
figure(1)

\% unit circle (plotted for reference)
cc = exp(1j*2*pi*(0:0.01:1));
ccx = A*real(cc);
cci = A*imag(cc);
%% Animation
for tt = 0:1/fs:dur
    tic; % establish time-reference
    plot(ccx, cci, ':', ...
         [0 A*cos(2*pi*ff*tt+phi)], [0 A*sin(2*pi*ff*tt+phi)], '-ob');
    axis('square')
    axis([-A A -A A]);
    title(TitleString)
    xlabel('Real')
    ylabel('Imag')
    grid on;

drawnow % force plots to be redrawn

    te = toc;

    % pause until the next sampling instant, if possible
    if ( te < 1/fs)
        pause(1/fs-te)
    end
end
Exercise

The continuous-time sinusoidal signal

\[ x(t) = \cos(2\pi t + \pi/4) + \cos(2\pi 3t + \pi/4) + \]
\[ \cos(2\pi 7t + \pi/4) + \cos(2\pi 8t + \pi/4) + \cos(2\pi 12t + \pi/4) \]

is sampled at the sampling rate \( f_s = 10 \text{ Hz} \).

1. Compute the spectrum of the above signal.

2. For each sinusoid in the above sum individually, determine if it is oversampled or undersampled and if folding or aliasing occurs.

3. For each sinusoid individually, determine the apparent frequency and the phase of the sampled signal.

4. Assume the sampled signal is reconstructed via an ideal D-C converter; write an expression for the resulting signal.
Lecture: The Sampling Theorem
The Sampling Theorem

- We have analyzed the relationship between the frequency $f$ of a sinusoid and the sampling rate $f_s$.
  - We saw that the ratio $f / f_s$ must be less than $1 / 2$, i.e., $f_s > 2 \cdot f$. Otherwise aliasing or folding occurs.
- This insight provides the first half of the famous sampling theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1 / T_s$ that is greater than $2 \cdot f_{\text{max}}$.

- This very import result is attributed to Claude Shannon and Harry Nyquist.
Reconstructing a Signal from Samples

- The sampling theorem suggests that the original continuous-time signal $x(t)$ can be recreated from its samples $x[n]$.
  - Assuming that samples were taken at a high enough rate.
  - This process is referred to as reconstruction or D-to-C conversion (discrete-time to continuous-time conversion).
- In principle, the continuous-time signal is reconstructed by placing a suitable pulse at each sample location and adding all pulses.
  - The amplitude of each pulse is given by the sample value.
Suitable Pulses

- Suitable pulses include
  - Rectangular pulse (zero-order hold):
    \[
    p(t) = \begin{cases} 
    1 & \text{for } -T_s/2 \leq t < T_s/2 \\
    0 & \text{else.}
    \end{cases}
    \]
  - Triangular pulse (linear interpolation)
    \[
    p(t) = \begin{cases} 
    1 + t/T_s & \text{for } -T_s \leq t \leq 0 \\
    1 - t/T_s & \text{for } 0 \leq t \leq T_s \\
    0 & \text{else.}
    \end{cases}
    \]
Reconstruction

- The reconstructed signal $\hat{x}(t)$ is computed from the samples and the pulse $p(t)$:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot p(t - nT_s).$$

- The reconstruction formula says:
  - place a pulse at each sampling instant ($p(t - nT_s)$),
  - scale each pulse to amplitude $x[n]$,
  - add all pulses to obtain the reconstructed signal.
Ideal Reconstruction

- Reconstruction with the above pulses will be pretty good.
  - Particularly, when the sampling rate is much greater than twice the signal frequency (significant oversampling).
- However, reconstruction is not perfect as suggested by the sampling theorem.
- To obtain perfect reconstruction the following pulse must be used:
  \[ p(t) = \frac{\sin(\pi t / T_s)}{\pi t / T_s}. \]
- This pulse is called the sinc pulse.
- Note, that it is of infinite duration and, therefore, is not practical.
  - In practice a truncated version may be used for excellent reconstruction.
The sinc pulse
Part V

Introduction to Linear, Time-Invariant Systems
Lecture: Introduction to Systems and FIR filters
A system is used to process an input signal $x[n]$ and produce the output signal $y[n]$.

We focus on discrete-time signals and systems;

a corresponding theory exists for continuous-time signals and systems.

Many different systems:

- Filters: remove undesired signal components,
- Modulators and demodulators,
- Detectors: determine if a specific signal is present.
Representative Examples

The following are examples of systems:

- **Squarer**: \( y[n] = (x[n])^2 \);
- **Modulator**: \( y[n] = x[n] \cdot \cos(2\pi f n) \);
- **Averager**: \( y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \);
- **FIR Filter**: \( y[n] = \sum_{k=0}^{M} b_k x[n-k] \)

In MATLAB, systems are generally modeled as functions with \( x[n] \) as the first input argument and \( y[n] \) as the output argument.

**Example**: first two lines of function implementing a squarer.

```matlab
function yy = squarer(xx)
% squarer - output signal is the square of the input signal
```
Systems

Special Signals

Linear, Time-invariant Systems

Convolutions and Linear, Time-Invariant Systems

Squarer

- System relationship between input and output signals:

\[ y[n] = (x[n])^2. \]

- **Example:** Input signal: \( x[n] = \{1, 2, 3, 4, 3, 2, 1\} \)
  - **Notation:** \( x[n] = \{1, 2, 3, 4, 3, 2, 1\} \) means
    \[ x[0] = 1, \quad x[1] = 2, \quad \ldots, \quad x[6] = 1; \]
    all other \( x[n] = 0. \)

- Output signal: \( y[n] = \{1, 4, 9, 16, 9, 4, 1\} \).
Modulator

- System relationship between input and output signals:

\[ y[n] = (x[n]) \cdot \cos(2\pi \hat{f} n); \]

where the modulator frequency \( \hat{f} \) is a parameter of the system.

- **Example:**
  - Input signal: \( x[n] = \{1, 2, 3, 4, 3, 2, 1\} \)
  - assume \( \hat{f} = 0.5 \), i.e., \( \cos(2\pi \hat{f} n) = \{\ldots, 1, -1, 1, -1, \ldots\} \).
  - Output signal: \( y[n] = \{1, -2, 3, -4, 3, -2, 1\} \).
Averager

- System relationship between input and output signals:

  \[ y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \]

  \[ = \frac{1}{M} \cdot (x[n] + x[n-1] + \ldots + x[n-(M-1)]) \]

  \[ = \sum_{k=0}^{M-1} \frac{1}{M} \cdot x[n-k]. \]

- This system computes the *sliding average* over the \( M \) most recent samples.

- **Example:** Input signal: \( x[n] = \{1, 2, 3, 4, 3, 2, 1\} \)

- For computing the output signal for finite-length inputs, a table is very useful.
  - synthetic multiplication table.
3-Point Averager \((M = 3)\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x[n])</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{1}{M} \cdot x[n])</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>(\frac{4}{3})</td>
<td>1</td>
<td>(\frac{2}{3})</td>
<td>(\frac{1}{3})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(+\frac{1}{M} \cdot x[n-1])</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>(\frac{4}{3})</td>
<td>1</td>
<td>(\frac{2}{3})</td>
<td>(\frac{1}{3})</td>
<td>0</td>
</tr>
<tr>
<td>(+\frac{1}{M} \cdot x[n-2])</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>(\frac{4}{3})</td>
<td>1</td>
<td>(\frac{2}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>(y[n])</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>(\frac{10}{3})</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

\[ y[n] = \{\frac{1}{3}, 1, 2, 3, \frac{10}{3}, 3, 2, 1, \frac{1}{3}\} \]
General FIR Filter

- The M-point averager is a special case of the general FIR filter.
  - FIR stands for Finite Impulse Response; we will see what this means later.
- The system relationship between the input $x[n]$ and the output $y[n]$ is given by
  \[ y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]. \]
  where $b_k$ are called the filter coefficients.
General FIR Filter

- System relationship:
  \[ y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]. \]

- The filter coefficients \( b_k \) determine the characteristics of the filter.
  - Much more on the relationship between the filter coefficients \( b_k \) and the characteristics of the filter later.

- Clearly, with \( b_k = \frac{1}{M} \) for \( k = 0, 1, \ldots, M - 1 \) we obtain the \( M \)-point averager.

- Again, for short input signals computation of the output signal can be done via a synthetic multiplication table.
  - Example: \( x[n] = \{1, 2, 3, 4, 3, 2, 1\} \) and \( b_k = \{1, -2, 1\} \).
**FIR Filter** \( (b_k = \{1, -2, 1\}) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x[n] )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
1 \cdot x[n] = 0 1 2 3 4 3 2 1 0 0 \\
-2 \cdot x[n-1] = 0 0 -2 -4 -6 -8 -6 -4 -2 0 \\
+1 \cdot x[n-2] = 0 0 0 1 2 3 4 3 2 1 \\

\[
y[n] = 0 1 0 0 0 -2 0 0 0 1
\]

- \( y[n] = \{1, 0, 0, 0, -2, 0, 0, 0, 1\} \)

- Note that the output signal \( y[n] \) is longer than the input signal \( x[n] \).
Exercise

1. Find the output signal $y[n]$ for an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

with filter coefficients $b_k = \{1, -1, 2\}$ when the input signal is $x[n] = \{1, 2, 4, 2, 4, 2, 1\}$. 
Unit Step Sequence and Unit Step Response

- The signal with samples

\[ u[n] = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0 \end{cases} \]

is called the unit-step sequence or unit-step signal.

- The output of an FIR filter when the input is the unit-step signal \( x[n] = u[n] \) is called the unit-step response \( r[n] \).
Unit-Step Response of the 3-Point Averager

- Input signal: \( x[n] = u[n] \).
- Output signal: \( r[n] = \frac{1}{3} \sum_{k=0}^{2} u[n - k] \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u[n] )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>( \frac{1}{3} u[n] )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>\ldots</td>
</tr>
<tr>
<td>( + \frac{1}{3} u[n - 1] )</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>\ldots</td>
</tr>
<tr>
<td>( + \frac{1}{3} u[n - 2] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>\ldots</td>
</tr>
<tr>
<td>( r[n] )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>1</td>
<td>1</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
Unit-Impulse Sequence and Unit-Impulse Response

- The signal with samples

\[ \delta[n] = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0 \end{cases} \]

is called the unit-impulse sequence or unit-impulse signal.

- The output of an FIR filter when the input is the unit-impulse signal \((x[n] = \delta[n])\) is called the unit-impulse response, denoted \(h[n]\).

- Typically, we will simply call the above signals simply impulse signal and impulse response.

- We will see that the impulse-response captures all characteristics of a FIR filter.
  - This implies that impulse response is a very important concept!
Unit-Impulse Response of a FIR Filter

- Input signal: \( x[n] = \delta[n] \).
- Output signal: \( h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \).

\[
\begin{array}{c|cccccccc}
 n & -1 & 0 & 1 & 2 & 3 & \ldots & M \\
 \delta[n] & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\
 \hline
 b_0 \cdot \delta[n] & 0 & b_0 & 0 & 0 & 0 & \ldots & 0 \\
 + b_1 \cdot \delta[n-1] & 0 & 0 & b_1 & 0 & 0 & \ldots & 0 \\
 + b_2 \cdot \delta[n-2] & 0 & 0 & 0 & b_2 & 0 & \ldots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
 + b_M \cdot \delta[n-M] & 0 & 0 & 0 & 0 & 0 & \ldots & b_M \\
 \hline
 h[n] & 0 & b_0 & b_1 & b_2 & b_3 & \ldots & b_M \\
\end{array}
\]
Important Insights

▶ For an FIR filter, the impulse response equals the sequence of filter coefficients:

\[ h[n] = \begin{cases} 
  b_n & \text{for } n = 0, 1, \ldots, M \\
  0 & \text{else.} 
\end{cases} \]

▶ Because of this relationship, the system relationship for an FIR filter can also be written as

\[ y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k] = \sum_{-\infty}^{\infty} h[k] x[n-k]. \]

▶ The operation \( y[n] = h[n] * x[n] = \sum_{-\infty}^{\infty} h[k] x[n-k] \) is called convolution; it is a very, very important operation.
Exercise

1. Find the impulse response $h[n]$ for the FIR filter with difference equation

   $$y[n] = 2 \cdot x[n] + x[n - 1] - 3 \cdot x[n - 3].$$

2. Compute the output signal, when the input signal is $x[n] = u[n]$.

3. Compute the output signal, when the input signal is $x[n] = a^{-n} \cdot u[n]$.
Lecture: Linear, Time-Invariant Systems
Introduction

- We have introduced systems as devices that process an input signal $x[n]$ to produce an output signal $y[n]$.

- **Example Systems:**
  - **Squarer:** $y[n] = (x[n])^2$
  - **Modulator:** $y[n] = x[n] \cdot \cos(2\pi \hat{f} n)$, with $0 < \hat{f} \leq \frac{1}{2}$.
  - **FIR Filter:**
    $$y[n] = \sum_{k=0}^{M} h[k] \cdot x[n - k].$$

Recall that $h[k]$ is the impulse response of the filter and that the above operation is called convolution of $h[n]$ and $x[n]$.

- **Objective:** Define important characteristics of systems and determine which systems possess these characteristics.
Causal Systems

- **Definition:** A system is called causal when it uses only the present and past samples of the input signal to compute the present value of the output signal.

- Causality is usually easy to determine from the system equation:
  - The output $y[n]$ must depend only on input samples $x[n], x[n - 1], x[n - 2], \ldots$
  - Input samples $x[n + 1], x[n + 2], \ldots$ must not be used to find $y[n]$.

- **Examples:**
  - All three systems on the previous slide are causal.
  - The following system is non-causal:

$$y[n] = \frac{1}{3} \sum_{k=-1}^{1} x[n - k] = \frac{1}{3} (x[n + 1] + x[n] + x[n - 1]).$$
Linear Systems

The following test procedure defines linearity and shows how one can determine if a system is linear:

1. **Reference Signals:** For $i = 1, 2$, pass input signal $x_i[n]$ through the system to obtain output $y_i[n]$.

2. **Linear Combination:** Form a new signal $x[n]$ from the linear combination of $x_1[n]$ and $x_2[n]$:

   $$x[n] = x_1[n] + x_2[n].$$

   Then, Pass signal $x[n]$ through the system and obtain $y[n]$.

3. **Check:** The system is linear if

   $$y[n] = y_1[n] + y_2[n]$$

   The above must hold for all inputs $x_1[n]$ and $x_2[n]$.

   For a linear system, the **superposition** principle holds.
These two outputs must be identical.
Example: Squarer

▶ Squarer: \( y[n] = (x[n])^2 \)

1. **References**: \( y_i[n] = (x_i[n])^2 \) for \( i = 1, 2 \).

2. **Linear Combination**: \( x[n] = x_1[n] + x_2[n] \) and

\[
\begin{align*}
    y[n] &= (x[n])^2 = (x_1[n] + x_2[n])^2 \\
         &= (x_1[n])^2 + (x_2[n])^2 + 2x_1[n]x_2[n].
\end{align*}
\]

3. **Check**:

\[
y[n] \neq y_1[n] + y_2[n] = (x_1[n])^2 + (x_2[n])^2.
\]

▶ **Conclusion**: not linear.
Example: Modulator

▶ Modulator: \( y[n] = x[n] \cdot \cos(2\pi \hat{f} n) \)

1. References: \( y_i[n] = x_i[n] \cdot \cos(2\pi \hat{f} n) \) for \( i = 1, 2 \).
2. Linear Combination: \( x[n] = x_1[n] + x_2[n] \) and

\[
\begin{align*}
y[n] &= x[n] \cdot \cos(2\pi \hat{f} n) \\
&= (x_1[n] + x_2[n]) \cdot \cos(2\pi \hat{f} n).
\end{align*}
\]

3. Check:

\[
y[n] = y_1[n] + y_2[n] = x_1[n] \cdot \cos(2\pi \hat{f} n) + x_2[n] \cdot \cos(2\pi \hat{f} n).
\]

▶ Conclusion: linear.
Example: FIR Filter

- **FIR Filter:** \( y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k] \)

1. **References:** \( y_i[n] = \sum_{k=0}^{M} h[k] \cdot x_i[n-k] \) for \( i = 1, 2 \).
2. **Linear Combination:** \( x[n] = x_1[n] + x_2[n] \) and

   \[
   y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k] = \sum_{k=0}^{M} h[k] \cdot (x_1[n-k] + x_2[n-k]).
   \]

3. **Check:**

   \[
   y[n] = y_1[n] + y_2[n] = \sum_{k=0}^{M} h[k] \cdot x_1[n-k] + \sum_{k=0}^{M} h[k] \cdot x_2[n-k].
   \]

- **Conclusion:** linear.
Time-invariance

The following test procedure defines time-invariance and shows how one can determine if a system is time-invariant:

1. **Reference:** Pass input signal $x[n]$ through the system to obtain output $y[n]$.
2. **Delayed Input:** Form the delayed signal $x_d[n] = x[n - n_0]$. Then, Pass signal $x_d[n]$ through the system and obtain $y_d[n]$.
3. **Check:** The system is time-invariant if

   $$y[n - n_0] = y_d[n]$$

The above must hold for all inputs $x[n]$ and all delays $n_0$.

**Interpretation:** A time-invariant system does not change, over time, the way it processes the input signal.
These two outputs must be identical.
Example: Squarer

- **Squarer**: $y[n] = (x[n])^2$
  1. **Reference**: $y[n] = (x[n])^2$.
  2. **Delayed Input**: $x_d[n] = x[n - n_0]$ and
     
     $$y_d[n] = (x_d[n])^2 = (x[n - n_0])^2.$$ 
  3. **Check**: 
     
     $$y[n - n_0] = (x[n - n_0])^2 = y_d[n].$$

- **Conclusion**: time-invariant.
Example: Modulator

- **Modulator:** $y[n] = x[n] \cdot \cos(2\pi\hat{f}n)$.
  1. Reference: $y[n] = x[n] \cdot \cos(2\pi\hat{f}n)$.
  2. Delayed Input: $x_d[n] = x[n - n_0]$ and

$$y_d[n] = x_d[n] \cdot \cos(2\pi\hat{f}n) = x[n - n_0] \cdot \cos(2\pi\hat{f}n).$$

3. Check:

$$y[n - n_0] = x[n - n_0] \cdot \cos(2\pi\hat{f}(n - n_0)) \neq y_d[n].$$

- **Conclusion:** not time-invariant.
Example: Modulator

- Alternatively, to show that the modulator is not time-invariant, we construct a counter-example.
- Let $x[n] = \{0, 1, 2, 3, \ldots\}$, i.e., $x[n] = n$, for $n \geq 0$.
- Also, let $\hat{f} = \frac{1}{2}$, so that

$$\cos(2\pi \hat{f} n) = \begin{cases} 
1 & \text{for } n \text{ even} \\
-1 & \text{for } n \text{ odd}
\end{cases}$$

- Then, $y[n] = x[n] \cdot \cos(2\pi \hat{f} n) = \{0, -1, 2, -3, \ldots\}$.
- With $n_0 = 1$, $x_\sigma[n] = x[n - 1] = \{0, 0, 1, 2, 3, \ldots\}$, we get $y_\sigma[n] = \{0, 0, 1, -2, 3, \ldots\}$.
- Clearly, $y_\sigma[n] \neq y[n - 1]$.
- not time-invariant
Example: FIR Filter

- **Reference:** \( y[n] = \sum_{k=0}^{M} h[k] \cdot x[n - k] \).

- **Delayed Input:** \( x_d[n] = x[n - n_0] \), and

\[
y_d[n] = \sum_{k=0}^{M} h[k] \cdot x_d[n - k] = \sum_{k=0}^{M} h[k] \cdot x[n - n_0 - k].
\]

- **Check:**

\[
y[n - n_0] = \sum_{k=0}^{M} h[k] \cdot x[n - n_0 - k] = y_d[n]
\]

- **time-invariant**
Exercise

Let $u[n]$ be the unit-step sequence (i.e., $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$, otherwise).

The system is a 3-point averager:

$$y[n] = \frac{1}{3}(x[n] + x[n - 1] + x[n - 2]).$$

1. Find the output $y_1[n]$ when the input $x_1[n] = u[n]$.
2. Find the output $y_2[n]$ when the input $x_2[n] = u[n - 2]$.
4. Use linearity and time-invariance to find $y_2[n]$ and $y_3[n]$ without convolution.
Lecture: Convolution and Linear, Time-Invariant Systems
Overview

- **Today:** a really important, somewhat challenging, class.
- **Key result:** for every linear, time-invariant system (LTI system) the output is obtained from input via convolution.
  - Convolution is a very important operation!
- **Prerequisites from previous classes:**
  - Impulse signal and impulse response,
  - convolution,
  - linearity, and
  - time-invariance.
Reminders: Convolution and Impulse Response

► **We learned so far:**

► For FIR filters, input-output relationship

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k]. \]

► If \( x[n] = \delta[n] \), then \( y[n] = h[n] \) is called the **impulse response** of the system.
  ► For FIR filters:

\[ h[n] = \begin{cases} 
  b_n & \text{for } 0 \leq n \leq M \\
  0 & \text{else.} 
\end{cases} \]

► **Convolution:** generalization of FIR filter; input-output relationship

\[ y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n - k] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n - k]. \]
Reminders: Linearity and Time-Invariance

▶ **Linearity:**
  - For arbitrary input signals $x_1[n]$ and $x_2[n]$, let the outputs be denoted $y_1[n]$ and $y_2[n]$.
  - Further, for the input signal $x[n] = x_1[n] + x_2[n]$, let the output signal be $y[n]$.
  - The system is **linear** if $y[n] = y_1[n] + y_2[n]$.

▶ **Time-Invariance:**
  - For an arbitrary input signal $x[n]$, let the output be $y[n]$.
  - For the delayed input $x_d[n] = x[n - n_0]$, let the output be $y_d[n]$.
  - The system is **time-invariant** if $y_d[n] = y[n - n_0]$.

▶ **Today:** For any linear, time-invariant system: input-output relationship is $y[n] = x[n] * h[n]$.
Preliminaries

- We need a few more facts and relationships for the impulse signal $\delta[n]$.
- To start, recall:
  - If input to a system is the impulse signal $\delta[n]$, then, the output is called the impulse response, and is denoted by $h[n]$.
- We will derive a method for expressing arbitrary signals $x[n]$ in terms of impulses.
Sifting with Impulses

- **Question:** What happens if we multiply a signal $x[n]$ with an impulse signal $\delta[n]$?

- Because

  $$\delta[n] = \begin{cases} 
  1 & \text{for } n = 0 \\
  0 & \text{else,}
  \end{cases}$$

- it follows that

  $$x[n] \cdot \delta[n] = x[0] \cdot \delta[n] = \begin{cases} 
  x[0] & \text{for } n = 0 \\
  0 & \text{else}
  \end{cases}$$

- Using $\delta[n]$, we “sift” the sample $x[0]$ out of $x[n]$. 
Illustration

\[ x[n] \cdot \delta[n] \]
Sifting with Impulses

▶ **Related Question:** What happens if we multiply a signal \( x[n] \) with a delayed impulse signal \( \delta[n - k] \)?

▶ Recall that \( \delta[n - k] \) is an impulse located at the \( k \)-th sampling instance:

\[
\delta[n - k] = \begin{cases} 
1 & \text{for } n = k \\
0 & \text{else}
\end{cases}
\]

▶ It follows that

\[
x[n] \cdot \delta[n - k] = x[k] \cdot \delta[n - k] = \begin{cases} 
x[k] & \text{for } n = k \\
0 & \text{else}
\end{cases}
\]

▶ Using \( \delta[n - k] \), we “sift” the sample \( x[k] \) out of \( x[n] \).
Illustration

\[ x[n] \]

\[ \delta[n] \]

\[ x[n] \cdot \delta[n-2] \]
Decomposing a Signal with Impulses

- **Question:** What happens if we combine (add) signals of the form $x[n] \cdot \delta[n - k]$?
- Specifically, what is

\[ \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - k] \]

- **Notice:** the above sum represents the convolution of $x[n]$ and $\delta[n]$, $\delta[n] * x[n]$. 

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## Decomposing a Signal with Impulses

### Table

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x[n]$</th>
<th>$\delta[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>-1</td>
<td>$x[-1]$</td>
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<tr>
<td>1</td>
<td>$x[1]$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$x[2]$</td>
<td>...</td>
</tr>
</tbody>
</table>

\[
\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] = x[-1] \cdot x[0] \cdot x[1] \cdot x[2] \cdot \ldots
\]
Decomposing a Signal with Impulses

- From these considerations we conclude that

\[ \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - k] = x[n]. \]

- Notice that this implies

\[ x[n] \ast \delta[n] = x[n]. \]

- We now have a way to write a signal \( x[n] \) as a sum of scaled and delayed impulses.

- Next, we exploit this relationship to derive our main result.
Applying Linearity and Time-Invariance

- We know already that input $\delta[n]$ produces output $h[n]$ (impulse response). We write:
  $$\delta[n] \mapsto h[n].$$

- For a time-invariant system: delayed input causes delayed output.
  $$\delta[n - k] \mapsto h[n - k].$$

- For a linear system: scaled input, causes scaled output.
  $$x[k] \cdot \delta[n - k] \mapsto x[k] \cdot h[n - k].$$

- For a linear system: sum of inputs, causes sum of outputs.
  $$\sum_k x[k] \cdot \delta[n - k] = x[n] \mapsto \sum_k x[k] \cdot h[n - k] = x[n] \ast h[n]$$
Derivation of the Convolution Sum

- Linearity: linear combination of input signals produces output equal to linear combination of individual outputs.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[-1] \cdot \delta[n + 1]$</td>
<td>$x[-1] \cdot h[n + 1]$</td>
</tr>
<tr>
<td>$x[0] \cdot \delta[n]$</td>
<td>$x[0] \cdot h[n]$</td>
</tr>
<tr>
<td>$x[1] \cdot \delta[n - 1]$</td>
<td>$x[1] \cdot h[n - 1]$</td>
</tr>
<tr>
<td>$x[2] \cdot \delta[n - 1]$</td>
<td>$x[2] \cdot h[n - 2]$</td>
</tr>
</tbody>
</table>

\[ \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - k] = x[n] \quad \mapsto \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n - k] \]
Summary and Conclusions

▶ We just derived the convolution sum formula:

\[ y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]. \]

▶ We only assumed that the system is linear and time-invariant.

▶ We can conclude: for any linear, time-invariant system, the output is the convolution of input and impulse response.

▶ Convolution and impulse response are enormously important concepts!
Identity System

- From our discussion, we can draw another conclusion.
- **Question:** How can we characterize a LTI system for which the output $y[n]$ is the same as the input $x[n]$.
  - Such a system is called the *identity system*.
- Specifically, we want the impulse response $h[n]$ of such a system.
- As always, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input.
- Since the output is the same as the input for an identity system, we find the impulse response of the identity system

\[ h[n] = \delta[n]. \]
Ideal Delay Systems

Closely Related Question: How can one characterize a LTI system for which the output $y[n]$ is a delayed version of the input $x[n]$:

$$y[n] = x[n - n_0]$$

where $n_0$ is the delay introduced by the system. Such a system is called an ideal delay system.

Again, we want the impulse response $h[n]$ of such a system.

As before, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input. Since the output is merely a delayed version of the input, we find

$$h[n] = \delta[n - n_0].$$
Part VI

Frequency Response
Lecture: Introduction to Frequency Response
Introduction

- We have discussed:
  - Sinusoidal and complex exponential signals,
  - Spectrum representation of signals:
    - arbitrary signals can be expressed as the sum of sinusoidal (or complex exponential) signals.
  - Linear, time-invariant systems.
- Next: complex exponential signals as input to linear, time-invariant systems.

\[ A \exp(j2\pi f_d n + \phi) \rightarrow \text{System} \rightarrow y[n] = ? \]
Example: 3-Point Averaging Filter

Consider the 3-point averager:

\[ y[n] = \frac{1}{3} \sum_{k=0}^{2} x[n-k] = \frac{1}{3} \cdot (x[n] + x[n-1] + x[n-2]). \]

Question: What is the output \( y[n] \) if the input is \( x[n] = \exp(j2\pi\hat{f}n) \)?

- Recall that \( \hat{f} \) is the normalized frequency \( f/f_s \); we are assuming the signal is oversampled, \( |\hat{f}| < \frac{1}{2} \).
- Initially, assume \( A = 1 \) and \( \phi = 0 \); generalization is easy.
Delayed Complex Exponentials

- The 3-point averager involves delayed versions of the input signal.
- We begin by assessing the impact the delay has on the complex exponential input signal.
- For
  \[ x[n] = \exp(j2\pi \hat{f}n) \]
  a delay by \( k \) samples leads to
  \[
  x[n - k] = \exp(j2\pi \hat{f}(n - k)) = e^{j(2\pi \hat{f}n - 2\pi \hat{f}k)} = e^{j2\pi \hat{f}n}. e^{-j2\pi \hat{f}k}
  \]
  \[
  = e^{j(2\pi \hat{f}n + \phi_k)} = e^{j2\pi \hat{f}n}. e^{j\phi_k}
  \]
  where \( \phi_k = -2\pi \hat{f}k \) is the phase shift induced by the \( k \) sample delay.
Average of Delayed Complex Exponentials

Now, the output signal $y[n]$ is the average of three delayed complex exponentials

$$y[n] = \frac{1}{3} \sum_{k=0}^{2} x[n - k] = \frac{1}{3} \sum_{k=0}^{2} e^{j(2\pi \hat{f} n - 2\pi \hat{f} k)}$$

This expression involves the sum of complex exponentials of the same frequency; the phasor addition rule applies:

$$y[n] = e^{j2\pi \hat{f} n} \cdot \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi \hat{f} k}.$$ 

Important Observation: The output signal is a complex exponential of the same frequency as the input signal.

- The amplitude and phase are different.
Frequency Response of the 3-Point Averager

The output signal $y[n]$ can be rewritten as:

$$y[n] = e^{j2\pi \hat{f} n} \cdot \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi \hat{f} k}$$

$$= e^{j2\pi \hat{f} n} \cdot H(\hat{f}).$$

where

$$H(\hat{f}) = \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi \hat{f} k}$$

$$= \frac{1}{3} \cdot (1 + e^{-j2\pi \hat{f}} + e^{-j2\pi 2\hat{f}})$$

$$= \frac{1}{3} \cdot e^{-j2\pi \hat{f}} (e^{j2\pi \hat{f}} + 1 + e^{-j2\pi \hat{f}})$$

$$= \frac{e^{-j2\pi \hat{f}}}{3} (1 + 2 \cos(2\pi \hat{f})).$$
From the above, we can conclude:

- If the input signal is of the form $x[n] = \exp(j2\pi\hat{f}n)$,
- then the output signal is of the form $y[n] = H(\hat{f}) \cdot \exp(j2\pi\hat{f}n)$.

The function $H(\hat{f})$ is called the **frequency response** of the system.

**Note:** If we know $H(\hat{f})$, we can easily compute the output signal in response to a complex exponential input signal.
Examples

- Recall:

\[ H(\hat{f}) = \frac{e^{-j2\pi \hat{f}}}{3} (1 + 2 \cos(2\pi \hat{f})) \]

- Let \( x[n] \) be a complex exponential with \( \hat{f} = 0 \).
  - Then, all samples of \( x[n] \) equal to one.

- The output signal \( y[n] \) also has all samples equal to one.

- For \( \hat{f} = 0 \), the frequency response \( H(0) = 1 \).

- And, the output \( y[n] \) is given by

\[ y[n] = H(0) \cdot \exp(j2\pi 0n), \]

i.e., all samples are equal to one.
Examples

- Let \( x[n] \) be a complex exponential with \( \hat{f} = \frac{1}{3} \).
  - Then, the samples of \( x[n] \) are the periodic repetition of \( \{ 1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2} \} \).
  - The 3-point average over three consecutive samples equals zero; therefore, \( y[n] = 0 \).
- For \( \hat{f} = \frac{1}{3} \), the frequency response \( H(\hat{f}) = 0 \).
- Consequently, the output \( y[n] \) is given by
  \[
  y[n] = H\left(\frac{1}{3}\right) \cdot \exp(j2\pi \frac{1}{3} n) = 0.
  \]

Thus, all output samples are equal to zero.
Introduction to Frequency Response

Frequency Response of LTI Systems

Plot of Frequency Response

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General Complex Exponential

- Let $x[n]$ be a complex exponential of the form $Ae^{j(2\pi \hat{f} n + \phi)}$.
  - This signal can be written as
    $$x[n] = X \cdot e^{j2\pi \hat{f} n},$$
    where $X = Ae^{j\phi}$ is the phasor of the signal.

- Then, the output $y[n]$ is given by
  $$y[n] = H(\hat{f}) \cdot X \cdot \exp(j2\pi \hat{f} n).$$

**Interpretation:** The output is a complex exponential of the same frequency $\hat{f}$

- The phasor for the output signal is the product $H(\hat{f}) \cdot X$. 
Exercise

Assume that the signal $x[n] = \exp(j2\pi \hat{f} n)$ is input to a 4-point averager.

1. Give a general expression for the output signal and identify the frequency response of the system.

2. Compute the output signals for the specific frequencies $\hat{f} = 0$, $\hat{f} = 1/4$, and $\hat{f} = 1/2$. 
Lecture: The Frequency Response of LTI Systems
Introduction

- We have demonstrated that for linear, time-invariant systems
  - the output signal $y[n]$
  - is the convolution of the input signal $x[n]$ and the impulse response $h[n]$.

\[
y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k]
\]

- **Question:** Find the output signal $y[n]$ when the input signal is $x[n] = A \exp(j(2\pi f n + \phi))$. 
Response to a Complex Exponential

▶ **Problem**: Find the output signal $y[n]$ when the input signal is $x[n] = A \exp(j(2\pi \hat{f} n + \phi))$.

▶ Output $y[n]$ is convolution of input and impulse response

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=0}^{M} h[k] \cdot x[n - k]$$
$$= \sum_{k=0}^{M} h[k] \cdot A \exp(j(2\pi \hat{f}(n - k) + \phi))$$
$$= A \exp(j(2\pi \hat{f} n + \phi)) \cdot \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi \hat{f} k)$$
$$= A \exp(j(2\pi \hat{f} n + \phi)) \cdot H(\hat{f})$$

▶ The term

$$H(\hat{f}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi \hat{f} k)$$

is called the **Frequency Response** of the system.
Interpreting the Frequency Response

The Frequency Response of an LTI system with impulse response $h[n]$ is

$$H(\hat{f}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi \hat{f} k)$$

**Observations:**

- The response of a LTI system to a complex exponential signal is a complex exponential signal of the same frequency.
  - Complex exponentials are eigenfunctions of LTI systems.
- When $x[n] = A \exp(j(2\pi \hat{f} n + \phi))$, then $y[n] = x[n] \cdot H(\hat{f})$.
  - This is true only for exponential input signals, including complex exponentials!
Interpreting the Frequency Response

Observations:

- \( H(\hat{f}) \) is best interpreted in polar coordinates:
  \[
  H(\hat{f}) = |H(\hat{f})| \cdot e^{j\angle H(\hat{f})}.
  \]

- Then, for \( x[n] = A \exp(j(2\pi \hat{f} n + \phi) \)
  \[
  y[n] = x[n] \cdot H(\hat{f})
  = A \exp(j(2\pi \hat{f} n + \phi) \cdot |H(\hat{f})| \cdot e^{j\angle H(\hat{f})}
  = (A \cdot |H(\hat{f})|) \cdot \exp(j(2\pi \hat{f} n + \phi + \angle H(\hat{f})))
  \]

- The amplitude of the resulting complex exponential is the product \( A \cdot |H(\hat{f})| \).
  - Therefore, \( |H(\hat{f})| \) is called the gain of the system.

- The phase of the resulting complex exponential is the sum \( \phi + \angle H(\hat{f}) \).
  - \( \angle H(\hat{f}) \) is called the phase of the system.
Example

Let $h[n] = \{1, -2, 1\}$.

Then,

$$H(\hat{f}) = \sum_{k=0}^{2} h[k] \cdot \exp(-j2\pi\hat{f}k)$$

$$= 1 - 2 \cdot \exp(-j2\pi\hat{f}) + 1 \cdot \exp(-j2\pi\hat{f}2)$$

$$= \exp(-j2\pi\hat{f}) \cdot (\exp(j2\pi\hat{f}) - 2 + \exp(-j2\pi\hat{f}))$$

$$= \exp(-j2\pi\hat{f}) \cdot (2 \cos(2\pi\hat{f}) - 2).$$

Gain: $|H(\hat{f})| = |(2 \cos(2\pi\hat{f}) - 2)$
Example
Example

- The filter with impulse response $h[n] = \{1, -2, 1\}$ is a **high-pass** filter.
  - It rejects sinusoids with frequencies near $\hat{\omega} = 0$,
  - and passes sinusoids with frequencies near $\hat{\omega} = \frac{1}{2}$

- Note how the function of this system is much easier to describe in terms of the frequency response $H(\hat{\omega})$ than in terms of the impulse response $h[n]$.

**Question:** Find the output signal when input equals $x[n] = 2 \exp(j2\pi 1/4n - \pi/2)$.

**Solution:**

$$H\left(\frac{1}{4}\right) = \exp(-j2\pi \frac{1}{4}) \cdot (2 \cos(2\pi \frac{1}{4}) - 2) = -2e^{-j\pi/2} = 2e^{j\pi/2}.$$  
Thus,

$$y[n] = 2e^{j\pi/2} \cdot x[n] = 4 \exp(j2\pi n/4).$$
Exercise

1. Find the Frequency Response $H(\hat{f})$ for the LTI system with impulse response $h[n] = \{1, -1, -1, 1\}$.

2. Find the output for the input signal $x[n] = 2 \exp(j(2\pi n/3 - \pi/4))$. 
Computing Frequency Response in MATLAB

```matlab
function HH = FreqResp( hh, ff )
    % FreqResp - compute frequency response of LTI system
    %
    % inputs:
    %  hh - vector of impulse response coefficients
    %  ff - vector of frequencies at which to evaluate frequency response
    %
    % output:
    %  HH - frequency response at frequencies in ff.
    %
    % Syntax:
    %  HH = FreqResp( hh, ff )

    HH = zeros( size(ff) );
    for kk = 1:length(hh)
        HH = HH + hh(kk) * exp(-j*2*pi*(kk-1)*ff);
    end
```
Part VII

Appendix: Complex Numbers and Complex Algebra
Lecture: Introduction to Complex Numbers
Why Complex Numbers?

- Complex numbers are closely related to sinusoids.
- They eliminate the need for trigonometry ...
- ... and replace it with simple algebra.
  - Complex algebra is really simple - this is not an oxymoron.
- Complex numbers can be represented as vectors.
  - Used to visualize the relationship between sinusoids.
The Basics

- Complex unity: \( j = \sqrt{-1} \).
- Complex numbers can be written as
  \[ z = x + j \cdot y. \]

This is called the rectangular or cartesian form.
- \( x \) is called the real part of \( z \): \( x = \text{Re}\{z\} \).
- \( y \) is called the imaginary part of \( z \): \( y = \text{Im}\{z\} \).
- \( z \) can be thought of a vector in a two-dimensional plane.
  - Coordinates are \( x \) and \( y \).
  - Coordinate system is called the complex plane.
Illustration - The Complex Plane
Euler’s Formulas

▶ Euler’s formula provides the connection between complex numbers and trigonometric functions.

\[ e^{j\phi} = \cos(\phi) + j \cdot \sin(\phi). \]

▶ Euler’s formula allows conversion between trigonometric functions and exponentials.
  ▶ Exponentials have simple algebraic rules for multiplication, division, and powers!
  ▶ Inverse Euler’s formulas:

\[
\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}
\]

\[
\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}
\]

These relationships are very important.
Polar Form

- Recall $z = x + j \cdot y$
- From the diagram it follows that
  \[ z = r \cos(\phi) + jr \sin(\phi). \]
- And by Euler’s relationship:
  \[ z = r \cdot (\cos(\phi) + j \sin(\phi)) = r \cdot e^{j\phi} \]
- This is called the polar form.
Converting from Polar to Cartesian Form

- Some problems are best solved in rectangular coordinates, while others are easier in polar form.
  - Need to convert between the two forms.
- A complex number polar form \( z = r \cdot e^{j\phi} \) is easily converted to cartesian form.

\[
z = r \cos(\phi) + jr \sin(\phi).
\]

- Example:

\[
4 \cdot e^{j\pi/3} = 4 \cdot \cos(\pi/3) + j \cdot 4 \cdot \sin(\pi/3)
\]
\[
= 4 \cdot \frac{1}{2} + j \cdot 4 \cdot \frac{\sqrt{3}}{2}
\]
\[
= 2 + j \cdot 2 \cdot \sqrt{3}.
\]
Converting from Cartesian to Polar Form

- A complex number \( z = x + jy \) in cartesian form is converted to polar form via

\[
r = \sqrt{x^2 + y^2}
\]

and

\[
\tan(\phi) = \frac{y}{x}.
\]

- The computation of the angle \( \phi \) requires some care.
- One must distinguish between the cases \( x < 0 \) and \( x > 0 \).

\[
\phi = \begin{cases} 
\arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\
\arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 
\end{cases}
\]

- If \( x = 0 \), \( \phi \) equals \( +\pi/2 \) or \( -\pi/2 \) depending on the sign of \( y \).
The Unit Circle in the Complex Plane

Good to know:

- \( e^{i2\pi} = 1 \)
- \( e^{i\pi} = -1 \)
- \( e^{i\pi/2} = j \)
- \( e^{-i\pi/2} = -j \)
- \( |e^{i\phi}| = 1 \) for all \( \phi \)
- \( \exp(j(\phi + 2\pi)) = e^{i\phi} \)
Exercise

- Convert to polar form
  1. \( z = 1 + j \)
  2. \( z = 3 \cdot j \)
  3. \( z = -1 - j \)

- Convert to cartesian form
  1. \( z = 3e^{-j{3\pi}/4} \)
Lecture: Complex Algebra
Introduction

- All normal rules of algebra apply to complex numbers!
- One thing to look for: $j \cdot j = -1$.
- Some operations are best carried out in rectangular coordinates.
  - Addition and subtraction
  - Multiplication and division aren’t very hard, either.
- Others are easier in polar coordinates.
  - Multiplication and division.
  - Powers and roots
- New operation: conjugate complex.
- A little more subtle: absolute value.
Conjugate Complex

The *conjugate complex* $z^*$ of a complex number $z$ has

- the same real part as $z$: $\text{Re}\{z\} = \text{Re}\{z^*\}$, and
- the opposite imaginary part: $\text{Im}\{z\} = -\text{Im}\{z^*\}$.

**Rectangular form:**

If $z = x + jy$ then $z^* = x - jy$.

**Polar form:**

If $z = r \cdot e^{j\phi}$ then $z^* = r \cdot e^{-j\phi}$.

- Note, $z$ and $z^*$ are mirror images of each other in the complex plane with respect to the real axis.
Illustration - Conjugate Complex

Complex Numbers

\[ z = r \sin \phi \]

\[ z^* = -r \sin \phi \]

\[ y = r \sin \phi \]

\[ -y = -r \sin \phi \]
Addition and Subtraction

- Addition and subtraction can only be done in rectangular form.
  - If the complex numbers to be added are in polar form, convert to rectangular form, first.
- Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$.
- **Addition:**
  
  $$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- **Subtraction:**
  
  $$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- Complex addition works like *vector addition*. 
Illustration - Complex Addition

Complex Numbers

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Multiplication

- Multiplication of complex numbers is possible in both polar and rectangular form.
- **Polar Form:** Let \( z_1 = r_1 \cdot e^{i\phi_1} \) and \( z_2 = r_2 \cdot e^{i\phi_2} \), then
  \[
  z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \exp(j(\phi_1 + \phi_2)).
  \]

- **Rectangular Form:** Let \( z_1 = x_1 + jy_1 \) and \( z_2 = x_2 + jy_2 \), then
  \[
  z_1 \cdot z_2 = (x_1 + jy_1) \cdot (x_2 + jy_2) = x_1x_2 + j^2y_1y_2 + jx_1y_2 + jx_2y_1 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1).
  \]

- Polar form provides more insight: multiplication involves rotation in the complex plane (because of \( \phi_1 + \phi_2 \)).
Complex Numbers

Absolute Value

- The absolute value of a complex number $z$ is defined as
  \[ |z| = \sqrt{z \cdot z^*}, \text{ thus, } |z|^2 = z \cdot z^*. \]

  - Note, $|z|$ and $|z|^2$ are real-valued.
  - In MATLAB, `abs(z)` computes $|z|$.

- **Polar Form:** Let $z = r \cdot e^{j\phi}$,
  \[ |z|^2 = r \cdot e^{j\phi} \cdot r \cdot e^{-j\phi} = r^2. \]
  - Hence, $|z| = r$.

- **Rectangular Form:** Let $z = x + jy$,
  \[ |z|^2 = (x + jy) \cdot (x - jy) \]
  \[ = x^2 - j^2y^2 - jxy + jxy \]
  \[ = x^2 + y^2. \]
Complex Numbers

Division

- **Polar Form:** Let $z_1 = r_1 \cdot e^{j\phi_1}$ and $z_2 = r_2 \cdot e^{j\phi_2}$, then
  \[
  \frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \exp(j(\phi_1 - \phi_2)).
  \]

- Closely related to multiplication
  \[
  \frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{z_1 z_2^*}{z_2 z_2^*}.
  \]

- **Rectangular Form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then
  \[
  \frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1) \cdot (x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{(x_1 x_2 + y_1 y_2) + j(-x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2}.
  \]
Exercises

For $z_1 = 3\sqrt{2} + j \cdot 3\sqrt{2}$ and $z_2 = 2e^{-j\pi/2}$, compute

1. $z_1 + z_2$,
2. $z_1 \cdot z_2$,
3. $z_1 / z_2$, and
4. $|z_1|$.

Give your results in polar or rectangular forms.
Lecture: Complex Algebra - Continued
Powers of Complex Numbers

- A complex number \( z \) is easily raised to the \( n \)-th power if \( z \) is in polar form.

- Specifically,

\[
z^n = (r \cdot e^{i\phi})^n = r^n \cdot e^{in\phi}
\]

- The magnitude \( r \) is raised to the \( n \)-th power
- The phase \( \phi \) is multiplied by \( n \).

- The above holds for arbitrary values of \( n \), including
  - \( n \) an integer (e.g., \( z^2 \)),
  - \( n \) a fraction (e.g., \( z^{1/2} = \sqrt{z} \))
  - \( n \) a negative number (e.g., \( z^{-1} = 1/z \))
  - \( n \) a complex number (e.g., \( z^j \))
Roots of Unity

- Quite often all complex numbers $z$ solving the following equation must be found
  \[ z^N = 1 = e^{j2\pi} = e^{j4\pi} = \ldots \]

- Here $N$ is an integer.
- There are $N$ different complex numbers solving this equation.
- The solutions have the form
  \[ z_n = e^{j2\pi n/N} \text{ for } n = 0, 1, 2, \ldots, N - 1. \]

- Note that $z_n^N = e^{j2\pi n} = 1$!
- The solutions are called the $N$-th roots of unity.
- In the complex plane, all solutions lie on the unit circle and are separated by angle $2\pi / N$. 
Roots of a Complex Number

- The more general problem is to find all solutions of the equation

\[ z^N = r \cdot e^{i\phi} \]

\[ = r \cdot e^{i(\phi + 2\pi)} = \ldots = r \cdot e^{i(\phi + 2\pi n)} \]

- In this case, the \( N \) solutions are given by

\[ z_n = r^{1/N} \cdot \exp\left(j\frac{\phi + 2\pi n}{N}\right) \text{ for } n = 0, 1, 2, \ldots, N - 1. \]
Example: Roots of a Complex Number

- **Example:** Find all solutions of $z^5 = -2$.
- **Solution:**
  - Note $-2 = 2e^{i\pi}$, i.e., $r = 2$ and $\phi = \pi$.
  - There are $N = 5$ solutions:
    - All have magnitude $\sqrt[5]{2}$.
    - The five angles are $\pi/5$, $3\pi/5$, $5\pi/5$, $7\pi/5$, $9\pi/5$. 
Roots of a Complex Number
Exercises

- Simplify:
  1. $(\sqrt{2} - \sqrt{2}j)^8$
  2. $(\sqrt{2} - \sqrt{2}j)^{-1}$

- Find all solutions of $z^4 = 1 + j$.

- Challenge:
  1. $j^j$
  2. $\cos(jx)$
Lecture: Trigonometric Identities via Complex Algebra
Two Ways to Express $\cos(\phi)$

- First relationship: $\cos(\phi) = \text{Re}\{e^{j\phi}\}$
- Second relationship (inverse Euler):
  \[
  \cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}.
  \]

- The first form is best suited as the starting point for problems involving the cosine or sine of a sum.
  - $\cos(\alpha + \beta)$

- The second form is best when products of sines and cosines are needed
  - $\cos(\alpha) \cdot \cos(\beta)$

- **Rule of thumb:** look to create products of exponentials.
Example

Show that \( \cos(x + y) \) equals \( \cos(x) \cos(y) - \sin(x) \sin(y) \):

Solution:

\[
\cos(x + y) = \text{Re}\{e^{j(x+y)}\} = \text{Re}\{e^{jx} \cdot e^{jy}\} \\
= \text{Re}\{(\cos(x) + j\sin(x)) \cdot (\cos(y) + j\sin(y))\} \\
= \text{Re}\{(\cos(x) \cos(y) - \sin(x) \sin(y)) + j(\cos(x) \sin(y) + \sin(x) \cos(y))\} \\
= \cos(x) \cos(y) - \sin(x) \sin(y).
\]
Example

► Show that \( \cos(x) \cos(y) \) equals
\[ \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y) : \]

► Solution:

\[
\cos(x) \cos(y) = \frac{e^{ix} + e^{-ix}}{2} \frac{e^{iy} + e^{-iy}}{2} = \frac{e^{i(x+y)} + e^{i(-x-y)} + e^{i(x-y)} + e^{i(-x+y)}}{4}
\]
\[
= \frac{e^{i(x+y)} + e^{-j(x+y)}}{4} + \frac{e^{i(x-y)} + e^{-j(x-y)}}{4}
\]
\[
= \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y).
\]
Exercises

Show that

\[ \cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) \]