Reading  Madhow: Section 3.2.

Problems

1. Let \( x \) and \( y \) be elements of a normed linear vector space.
   (a) Determine whether the following are valid inner products for the indicated space.
   i. \( \langle x, y \rangle = x^T Ay \), where \( A \) is a nonsingular, \( N \times N \) matrix and \( x, y \) are elements of the space of \( N \)-dimensional vectors.
   ii. \( \langle x, y \rangle = xy^T \), where \( x \) and \( y \) are elements of the space of \( N \)-dimensional (column!) vectors.
   iii. \( \langle x, y \rangle = \int_0^T x(t)y(T - t) \, dt \), where \( x \) and \( y \) are finite energy signals defined over \([0, T]\).
   iv. \( \langle x, y \rangle = \int_0^T w(t)x(t)y(t) \, dt \), where \( x \) and \( y \) are finite energy signals defined over \([0, T]\) and \( w(t) \) is a non-negative function.
   v. \( E[XY] \), where \( X \) and \( Y \) are real-valued random variables having finite mean-square values.
   vi. \( \text{Cov}(X, Y) \), the covariance of the real-valued random variables \( X \) and \( Y \). Assume that \( X \) and \( Y \) have finite mean-square values.
   (b) Under what conditions is \( \int_0^T \int_0^T Q(t,u)x(t)y(u) \, dt \, du \) a valid inner product for the space of finite-energy functions defined over \([0, T]\)?

2. Let \( x(t) \) be a signal of finite energy over the interval \([0, T]\). In other words, \( x(t) \) is a vector in the Hilbert space \( L^2(0, T) \). Signals may be complex values, so that the appropriate inner product is
   \[ \langle x, y \rangle = \int_0^T x(t) \cdot y^*(t) \, dt. \]
   Consider subspace \( \mathcal{L} \) of \( L^2(0, T) \) that consists of signals of the form
   \( y_n(t) = X_n \exp(j2\pi nt/T) \) for \( 0 \leq t \leq T \),
   where \( X_n \) may be complex valued.
   (a) Find the signal \( \hat{y}_n(t) \) that best approximates the signal \( x(t) \), i.e., \( \hat{y}_n(t) \) minimizes \( \| x - y_n \| \) among all elements of \( \mathcal{L} \).
   \( \text{Hint:} \) Find the best complex amplitude \( \hat{X}_n \).
(b) Now define the error signal \( z(t) = x(t) - \hat{y}_n(t) \). Show that \( z(t) \) is orthogonal to the subspace \( \mathcal{L} \), i.e., it is orthogonal to all elements of \( \mathcal{L} \).

(c) How do the above results illustrate the projection theorem?

3. **Linear Regression**

The elements of a vector of random variables \( \vec{Y} \) follow the model

\[
Y_n = ax_n + b + N_n
\]

where \( x_n \) are known and \( N_n \) are zero mean, iid Gaussian noise samples with variance \( \sigma^2 \). The parameters \( a \) and \( b \) are to be determined. We can think of the solution to this problem as the projection of \( \vec{Y} \) onto the subspace spanned by \( a\vec{x} + b \).

(a) Determine the **least-squares estimates** for \( a \) and \( b \), i.e., find

\[
\hat{a}, \hat{b} = \arg \min_{a,b} \| \vec{Y} - (a\vec{x} + b) \|^2.
\]

(b) What are the expected values of these estimates, \( E[\hat{a}] \) and \( E[\hat{b}] \)?

(c) Compute \( \hat{a} \) and \( \hat{b} \), when data are given by the \( (x_n, Y_n) \) pairs

\[
\{(x_n, Y_n)\}_{n=1}^5 = \{(0, 1.3), (1, 0.2), (2, 0.1), (3, -0.4), (4, -1.2)\}.
\]

(d) Is it true that the least-squares estimates for \( a \) and \( b \) are given by the inner products

\[
\hat{a} = \langle \vec{Y}, \vec{x} \rangle \text{ and } \hat{b} = \langle \vec{Y}, \vec{1} \rangle?
\]

\( \vec{1} \) denotes a vector of 1’s. Explain why or why not?