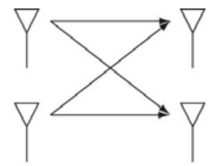
7. MIMO I: Spatial Multiplexing and Channel Modeling

Main Story

- So far we have only considered single-input multi-output (SIMO) and multi-input single-output (MISO) channels.
- They provide diversity and power gains but no degree-of-freedom (d.o.f.) gain.
- D.o.f gain is most useful in the high SNR regime.
- MIMO channels have a potential to provide d.o.f gain.
- We would like to understand how the d.o.f gain depends on the physical environment and come up with statistical models that capture the properties succinctly.
- We start with deterministic models and then progress to statistical ones.

MIMO Capacity via SVD

Narrowband MIMO channel:



$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$

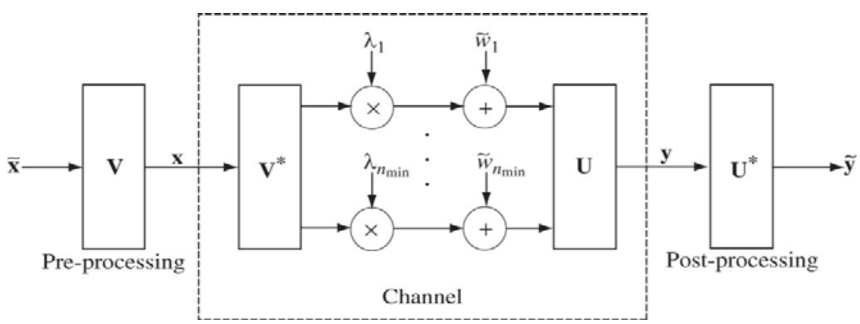
$\mathbf{H}$ is n_r by n_t, fixed channel matrix.

Singular value decomposition:

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$$

$\mathbf{U}$, $\mathbf{V}$ are complex orthogonal matrices and $\mathbf{\Lambda}$ real diagonal (singular values).

Spatial Parallel Channel



Capacity is achieved by waterfilling over the eigenmodes of $\mathbf{H}$. (Analogy to frequency-selective channels.)

Rank and Condition Number

At high SNR, equal power allocation is optimal:

$$C \approx \sum_{i=1}^{k} \log\left(1+\frac{P\lambda_i^2}{kN_0}\right) \approx k \log \mathsf{SNR} + \sum_{i=1}^{k} \log\left(\frac{\lambda_i^2}{k}\right)$$

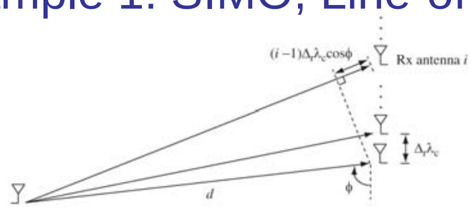
where k is the number of nonzero λ_i^2 's, i.e. the rank of $\mathbf{H}$.

The closer the condition number:

$$\frac{\max_i \lambda_i}{\min_i \lambda_i}$$

to 1, the higher the capacity.

Example 1: SIMO, Line-of-sight



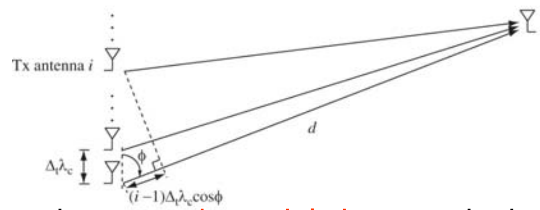
$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

$\mathbf{h}$ is along the receive spatial signature in the direction $\Omega := \cos\phi$:

$$\mathbf{e}_r(\Omega) := \frac{1}{\sqrt{n_r}} \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r\Omega) \\ \exp(-j2\pi 2\Delta_r\Omega) \\ \cdot \\ \cdot \\ \exp(-j2\pi(n_r-1)\Delta_r\Omega) \end{bmatrix}$$

n_r –fold power gain.

Example 2: MISO, Line-of-Sight

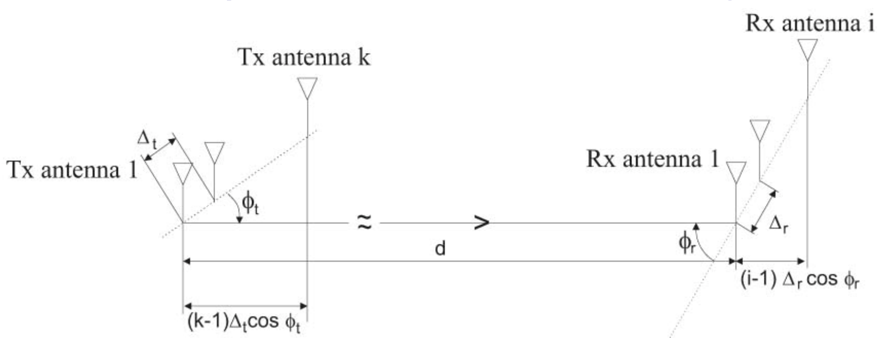


$\mathbf{h}$ is along the transmit spatial signature in the direction $\Omega := \cos\phi$:

$$\mathbf{e}_t(\Omega) := \frac{1}{\sqrt{n_t}}\begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_t\Omega) \\ \exp(-j2\pi 2\Delta_t\Omega) \\ \cdot \\ \cdot \\ \exp(-j2\pi(n_t-1)\Delta_t\Omega) \end{bmatrix}$$

n_t – fold power gain.

Example 3: MIMO, Line-of-Sight



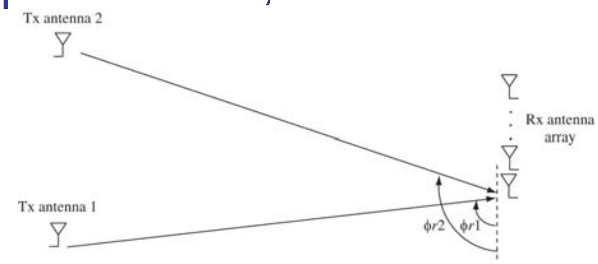
$$\mathbf{H} = G \cdot \mathbf{e}_r(\Omega_r)\,\mathbf{e}_t(\Omega_t)^*$$

n_r n_t – fold power gain

Rank 1, only one degree of freedom.

No spatial multiplexing gain.

Example 4: MIMO, Tx Antennas Apart



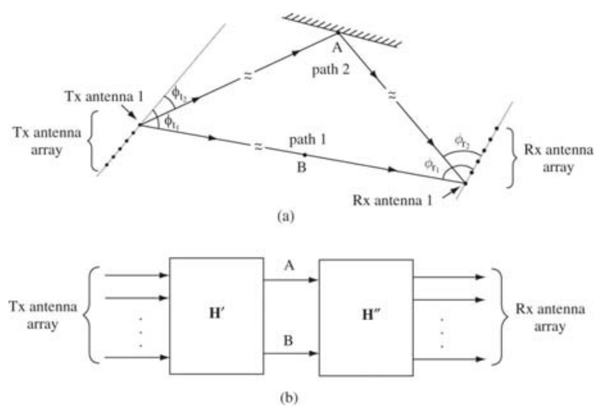
$$\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$$

$\mathbf{h}_i$ is the receive spatial signature from Tx antenna i along direction $\Omega_i = \cos \phi_{ri}$:

$$\mathbf{h}_i = G_i \cdot \mathbf{e}_\mathrm{r}(\Omega_i) .$$

Two degrees of freedom if h_1 and h_2 are different.

Example 5: Two-Path MIMO

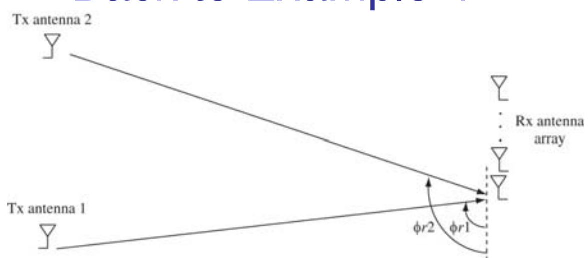


A scattering environment provides multiple degrees of freedom even when the antennas are close together.

Rank and Conditioning

- Question: Does spatial multiplexing gain increase without bound as the number of multipaths increase?
- The rank of H increases but looking at the rank by itself is not enough.
- The condition number matters.
- As the angular separation of the paths decreases, the condition number gets worse.

Back to Example 4



$$\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$$

$\mathbf{h}_i$ is the receive spatial signature from Tx antenna i along direction $\Omega_i = \cos \phi_{ri}$:

$$\mathbf{h}_i = G_i \cdot \mathbf{e}_\mathrm{r}(\Omega_i).$$

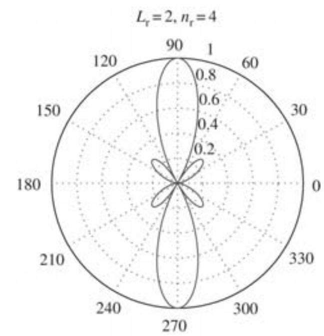
Condition number depends on $|\mathbf{e}_\mathrm{r}(\Omega_1)^* \mathbf{e}_\mathrm{r}(\Omega_2)|$.

Beamforming Patterns

The receive beamforming pattern associated with $\mathbf{e}_r(\Omega_0)$:

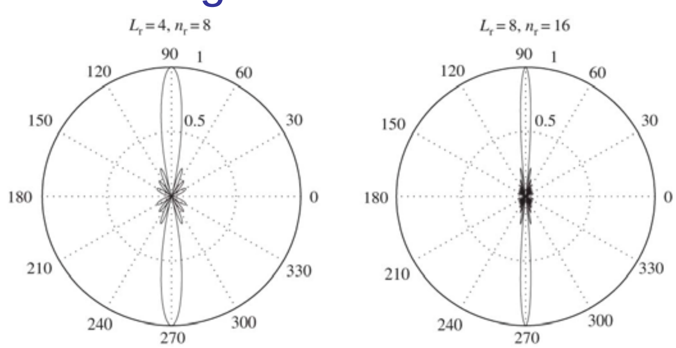
$$B_r(\Omega) := |\mathbf{e}_r(\Omega_0)^* \mathbf{e}_r(\Omega)|$$

L_r is the length of the antenna array, normalized to the carrier wavelength.



- Beamforming pattern gives the antenna gain in different directions.
- But it also tells us about angular resolvability.

Angular Resolution

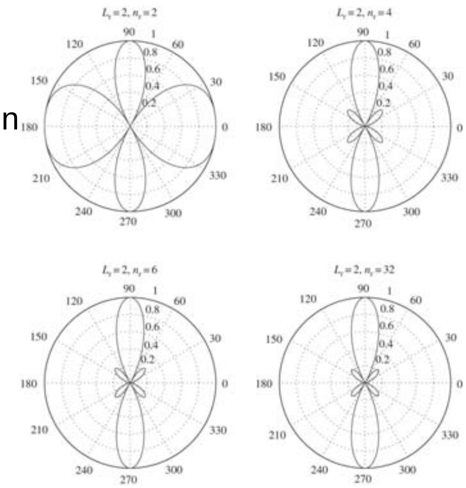


Antenna array of length L_r provides angular resolution of $1/L_r$: paths that arrive at angles closer is not very distinguishable.

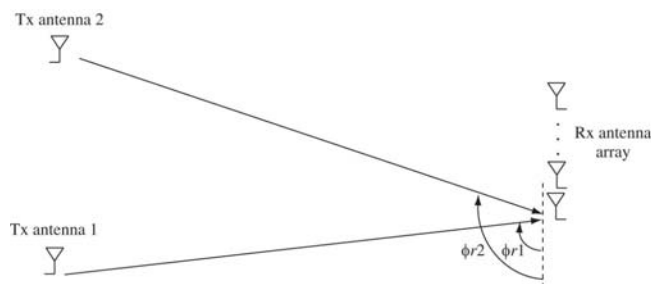
Varying Antenna Separation

Decreasing antenna separation beyond λ/2 has no impact on angular resolvability.

Assume λ/2 separation from now on (so n=2L).



Back to Example 4



Channel $\mathbf{H}$ is well conditioned if

$$|\Omega_1 - \Omega_2| \gg \frac{1}{L_r}.$$

i.e. the signals from the two Tx antennas can be resolved.

MIMO Channel Modeling

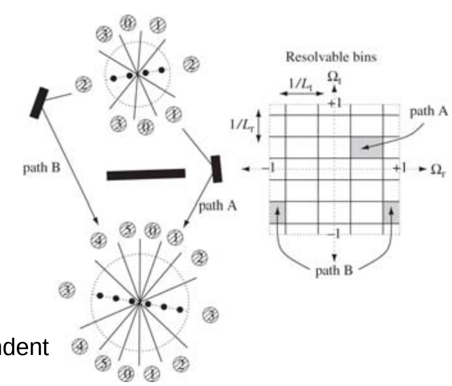
- Recall how we modeled multipath channels yesterday.
- Start with a deterministic continuous-time model.
- Sample to get a discrete-time tap delay line model.
- The physical paths are grouped into delay bins of width 1/W seconds, one for each tap.
- Each tap gain h_l is an aggregation of several physical paths and can be modeled as Gaussian.
- We can follow the same approach for MIMO channels.

MIMO Modeling in Angular Domain

The outgoing paths are grouped into resolvable bins $\{\mathcal{T}_l\}$ of angular width $1/L_t$

The incoming paths are grouped into resolvable bins $\{\mathcal{R}_k\}$ of angular width $1/L_r$.

The $(k,l)^{th}$ entry of $\mathbf{H}^a$ is (approximately) the aggregation of paths in $\mathcal{T}_l \cap \mathcal{R}_k$.



Can statistically model each entry as independent and Gaussian.

Bins that have no paths will have zero entries in $\mathbf{H}^a$.

Spatial-Angular Domain Transformation

What is the relationship between angular $\mathbf{H^a}$ and spatial $\mathbf{H}$?

$2L_t$ £ $2L_t$ transmit angular basis matrix (orthonormal):

$$\mathbf{U_t} := \left[\mathbf{e_t}(0), \mathbf{e_t}\left(\frac{1}{L_t}\right), \ldots, \mathbf{e_t}\left(2 - \frac{1}{L_t}\right)\right]$$

$2L_r$ £ $2L_r$ receive angular basis matrix (orthonormal):

$$\mathbf{U_r} := \left[\mathbf{e_r}(0), \mathbf{e_r}\left(\frac{1}{L_r}\right), \ldots, \mathbf{e_r}\left(2 - \frac{1}{L_r}\right)\right]$$

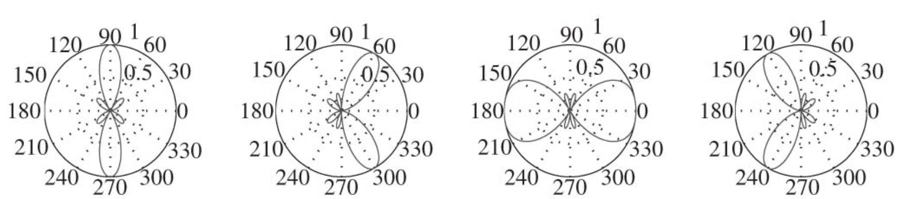
Input,output in angular domain:

$$\mathbf{x} = \mathbf{U_t}\mathbf{x^a}, \qquad \mathbf{y} = \mathbf{U_r}\mathbf{y^a}$$

so

$$\mathbf{H^a} := \mathbf{U}_r^*\mathbf{H}\mathbf{U}_t$$

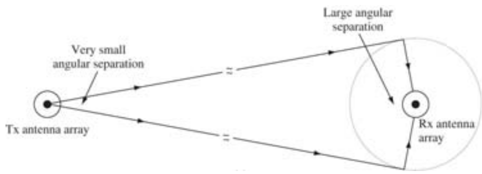
Angular Basis

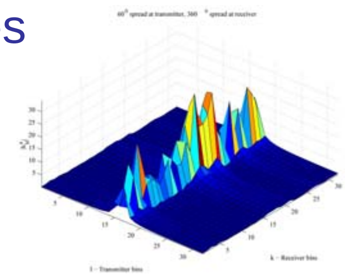


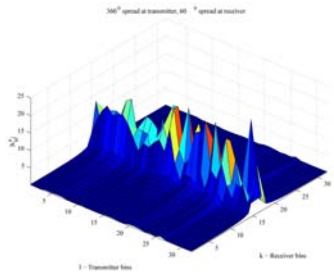
(a) $L_r = 2$, $n_r = 4$

- The angular transformation decomposes the received (transmit) signals into components arriving (leaving) in different directions.

Examples

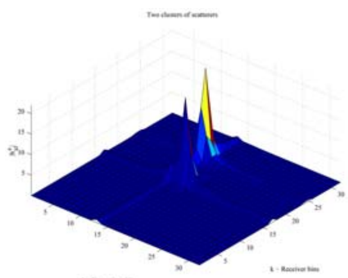


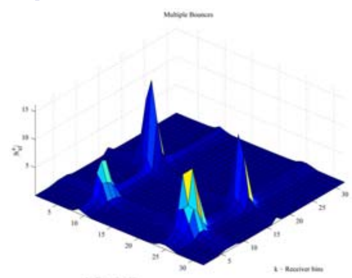


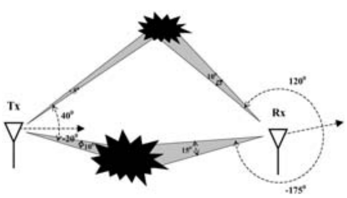


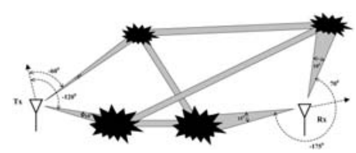
7: MIMO I: Spatial Multiplexing and Channel Modeling
Wireless Foundations

More Examples

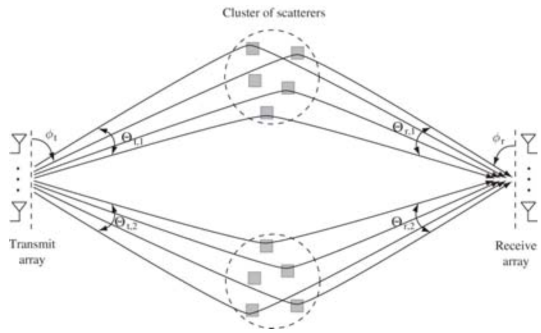






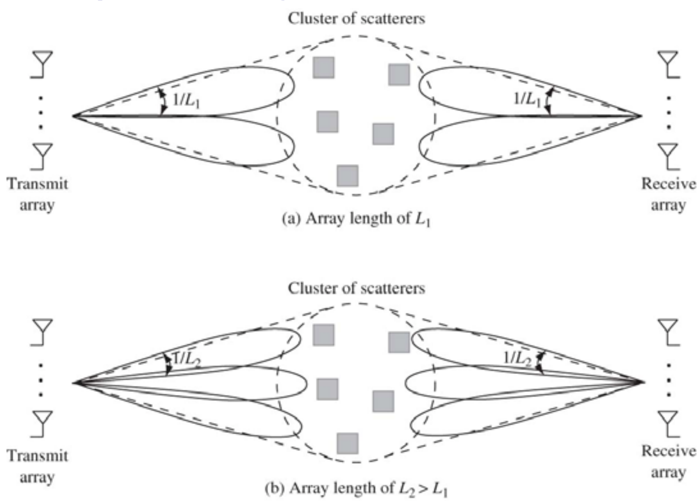


Clustered Model

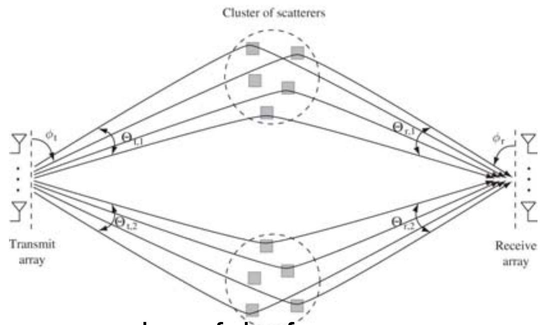


How many degrees of freedom are there in this channel?

Dependency on Antenna Size



Clustered Model



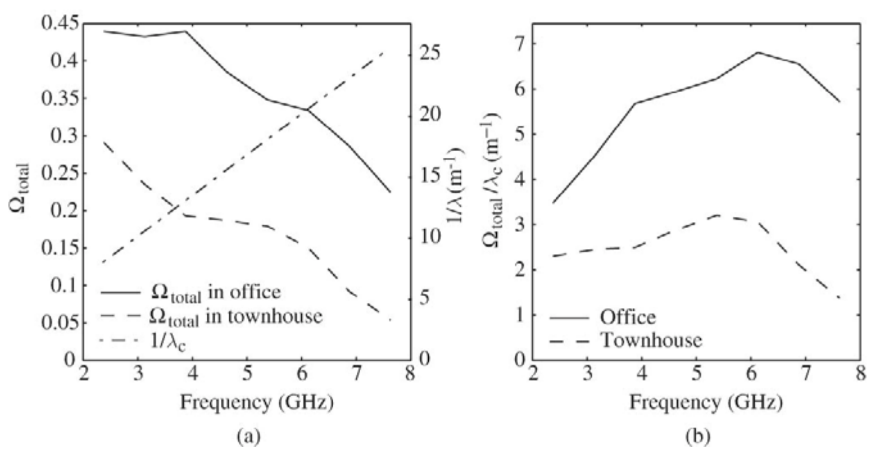
For L_t,L_r large, number of d.o.f.:

$$\min\{L_t\Omega_t, L_r\Omega_r\}$$

where

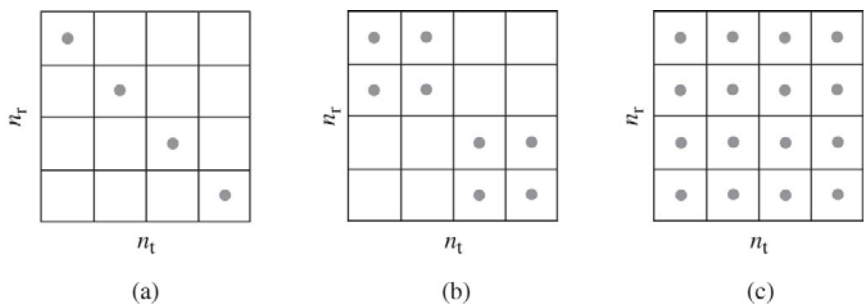
$$\Omega_t = \sum_i \Theta_{t,i}, \qquad \Omega_r = \sum_i \Theta_{r,i}$$

Dependency on Carrier Frequency



Measurements by Poon and Ho 2003.

Diversity and Dof



I.I.D. Rayleigh Model

Scatterers at all angles from
Tx and Rx.

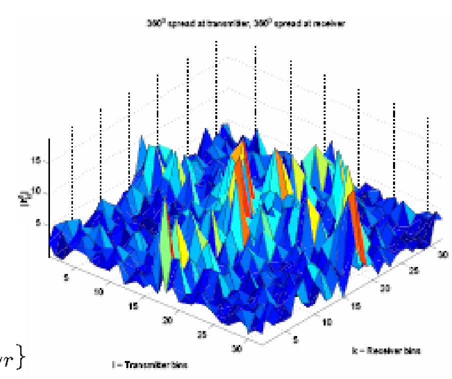
$$\mathbf{H} = \mathbf{U}_r \mathbf{H}^{\mathbf{a}} \mathbf{U}_t^*$$

$\mathbf{H}^{a}$ i.i.d. Rayleigh $ $\mathbf{H}$ i.i.d. Rayleigh

Angular spread

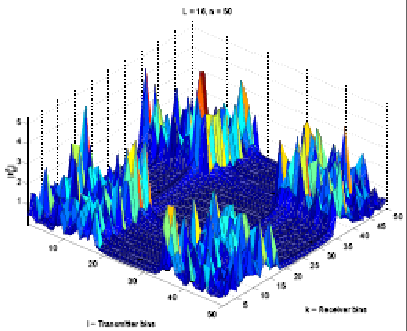
$$\Omega_t = \Omega_r = 2$$

$$\begin{aligned} \text{d.o.f.} &= \min\{2L_t, 2L_r\} \\ &= \min\{n_t, n_r\} \end{aligned}$$



Correlated Fading

- When scattering only comes from certain angles, H^a has zero entries.
- Corresponding spatial H has correlated entries.
- Same happens when antenna separation is less than $\lambda/2$ (but can be reduced to a lower-dimensional i.i.d. matrix)
- Angular domain model provides a physical explanation of correlation.



Analogy with Time-Frequency Channel Modeling

	Time-Frequency	Spatial-Angular
Domains	Time ↔ Frequency ↔	Angular Spatial
Resources	signal duration T bandwidth W	angular spreads Ω_t, Ω_r antenna array lengths L_t,L_r
Resolution of multipaths	into delay bins of $1/W$	into angular bins of $1/L_t$ by $1/L_r$
d.o.f.	WT	$\min(L_t\Omega_t, L_r\Omega_r)$
Diversity	# of non-zero delay bins	# of non-zero angular bins