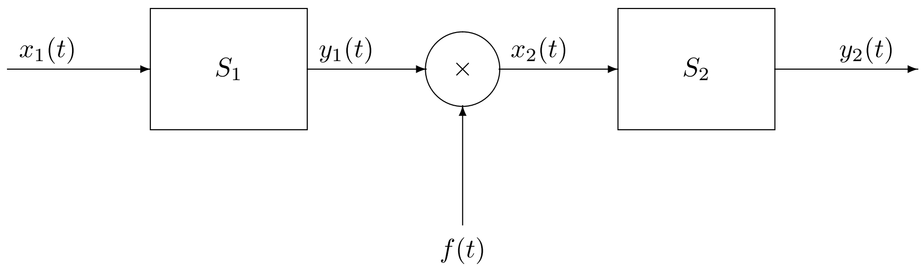
ECE 460: Communication and Information Theory
Practice Problems I

The following are old exam problem that may help you prepare for the midterm exam.

1. Consider the following system



The spectrum of the input signal $x_1(t)$ is

$$X_1(f) = \begin{cases} 1 & \text{for } f \leq f_0 \\ 0 & \text{for } f > f_0 \end{cases}$$

and the transfer function of the linear system S_1 is given by

$$H_1(f) = \begin{cases} 1 - \frac{|f|}{2f_0} & \text{for } f \leq 2f_0 \\ 0 & \text{for } f > 2f_0 \end{cases}$$

(a) Compute and sketch the spectrum of the signal $y_1(t)$.

(b) By multiplying $y_1(t)$ with the pulse train

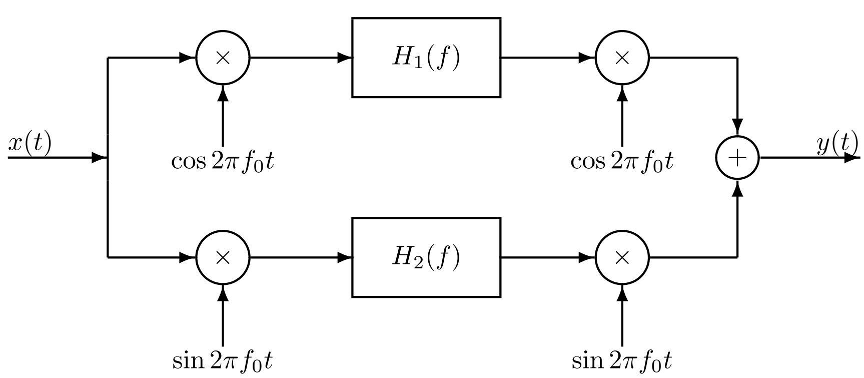
$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

we generate the signal $x_2(t)$. Find a condition on T such that $y_1(t)$ can be reconstructed completely.

(c) Assume now that $T > \frac{2}{f_0}$. Compute and sketch the spectrum of $x_2(t)$.

(d) The system S_2 is used to recover the input signal $x_1(t)$. Find the transfer function $H_2(f)$ of S_2 such that $y_2(t) = x_1(t)$.

2. **Linear Systems** (40 points)
Consider the following system in which $H_1(f)$ and $H_2(f)$ are transfer functions of linear systems with impulse response $h_1(t)$ and $h_2(t)$, respectively.



(a) Find the output $y(t)$ when $x(t) = \delta(t)$ is the input. Repeat for $x(t) = \delta(t - \frac{1}{4f_0})$.

(b) Now, find $y(t)$ for $x(t) = \delta(t - \tau)$, where $\tau > 0$ is arbitrary.

(c) For what condition on $h_1(t)$ and $h_2(t)$ is the above system *time-invariant*?

(d) Assume that the two linear systems are identical with frequency response $H_1(f) = H_2(f) = \Pi(\frac{f}{f_1})$, where $f_1 < f_0$ and $\Pi(\cdot)$ denotes the rectangular pulse function, i.e., $\Pi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\Pi(x) = 0$ for $|x| > \frac{1}{2}$. Find the frequency response $H(f)$ of the overall system, i.e., the system with input $x(t)$ and output $y(t)$.

(e) Sketch the magnitude of $H(f)$.

Hint: You may need $\cos(a - b) = \cos a \cos b + \sin a \sin b$.

3. **Amplitude Modulation**
Consider the following amplitude modulated signal,

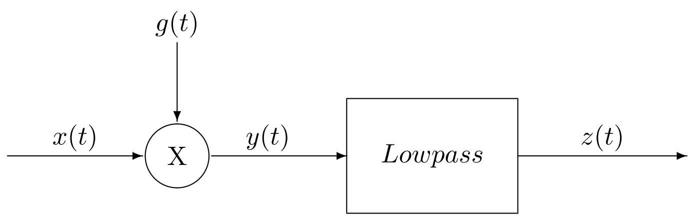
$$x(t) = (A + m(t)) \cdot \sin(2\pi f_c t).$$

The constant A has been chosen such that $(A + m(t)) > 0$ for all t. Furthermore, the spectrum of the message signal $m(t)$ is

$$M(f) = \Pi(f/2f_m) = \begin{cases} 1 & \text{for } |f| < f_m \\ 0 & \text{otherwise.} \end{cases}$$

The carrier frequency f_c is much larger than f_m.

To demodulate $x(t)$ we use the following system:



where the signal $g(t)$ is given by:

$$g(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2f_c} \\ -1 & \frac{1}{2f_c} \leq t < \frac{1}{f_c} \end{cases}$$

and $g(t) = g(t + 1/f_c)$.

(a) Sketch the spectrum $X(f)$ of the signal $x(t)$.

(b) Show that the signal $g(t)$ can be represented as a Fourier series by

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} \frac{1 - e^{-j\pi n}}{j\pi n} \exp(-j2\pi n f_c t) \\ &= \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin(2\pi(2n+1) f_c t). \end{aligned}$$

(c) Give an expression for the spectrum $G(f)$ of $g(t)$ and sketch $G(f)$.

(d) Using the results from parts (b) and (c), sketch and accurately label the spectrum $Y(f)$ of the signal $y(t) = x(t)g(t)$.

(e) How would you choose the cut-off frequency of the ideal lowpass, in order to recover the message signal $m(t)$?

4. **Single Sideband Signals**

(a) Verify that the inverse Fourier transform of

$$G(f) = \begin{cases} e^{-\alpha f} & \text{for } f \geq 0 \\ e^{\alpha f} & \text{for } f < 0 \end{cases}$$

is given by

$$g(t) = \frac{j4\pi t}{\alpha^2 + (2\pi t)^2}.$$

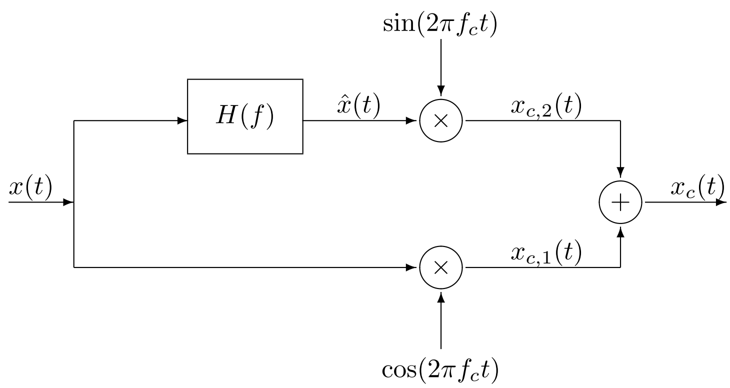
(b) By considering the limit as α tends to 0 of the result in part (a), show that the following is a Fourier transform pair

$$\frac{1}{\pi t} \leftrightarrow -j \cdot \text{sign}(f) = \begin{cases} -j & \text{for } f > 0 \\ 0 & \text{for } f = 0 \\ j & \text{for } f < 0. \end{cases}$$

(c) For the remainder of the problem, the signal $x(t)$ has the Fourier transform

$$X(f) = \begin{cases} 1 - \frac{|f|}{f_M} & \text{for } |f| \leq f_M \\ 0 & \text{else.} \end{cases}$$

Let $x(t)$ be input to the following system in which $H(f) = -j \cdot \text{sign}(f)$:



Derive an expression for the Fourier transform $\hat{X}(f)$ of the signal $\hat{x}(t)$ and sketch $\hat{X}(f)$.

(d) Assuming that f_c is much larger than f_M, compute expressions for the Fourier transforms of the signals $x_{c,1}(t)$, $x_{c,2}(t)$, and $x_c(t)$. Graph and accurately label these Fourier transforms.

(e) Indicate an application where the above system would be useful.

5. **Amplitude Modulation**
Consider the following amplitude modulated signal,

$$x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

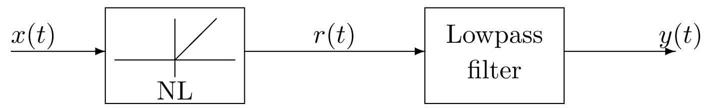
The constant A has been chosen such that $(A + m(t)) > 0$ for all t. Furthermore, the spectrum of the message signal $m(t)$ is bandlimited, such that

$$M(f) = 0 \quad \text{for } |f| > f_m$$

The carrier frequency f_c is much larger than f_m.

(a) For a typical spectrum $M(f)$, sketch the magnitude of the spectrum $X(f)$ of the signal $x(t)$.

(b) The envelope detector discussed in class can be modeled as a nonlinear device followed by a lowpass filter as shown in the following block diagram:



The nonlinear device NL is described by the relationship between its input $x(t)$ and its output $r(t)$ as

$$r(t) = \mathrm{NL}(x(t)) = \begin{cases} x(t) & \text{if } x(t) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume the signal $g(t) = \cos(2\pi f_c t)$ is passed through the above nonlinearity. Sketch the resulting signal $\mathrm{NL}(g(t))$ and show that this signal can be represented as a Fourier series by

$$\mathrm{NL}(g(t)) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{\sin(\frac{\pi}{2}(n-1))}{n-1} + \frac{\sin(\frac{\pi}{2}(n+1))}{n+1} \right) \mathrm{e}^{-jn2\pi f_c t}.$$

(c) For a typical message signal $m(t)$, sketch the signals $x(t)$ and $r(t)$. Further, sketch the product of the signals $(A + m(t))$ and $\mathrm{NL}(g(t))$ to verify that $r(t) = (A + m(t)) \cdot \mathrm{NL}(g(t))$.

(d) Using the results from parts (b) and (c), compute the Fourier transform of the signal $r(t)$. Sketch and accurately label the magnitude of the Fourier transform of $r(t)$.

(e) How would you choose the cutoff frequency of the lowpass filter in the demodulator above?

6. Throughout this problem consider the following two signals

$$s_1(t) = \cos(2\pi f_0 t) \cdot \Pi(t/T_0)$$

and

$$s_2(t) = (f_1 + f_2) \cdot \mathrm{sinc}(\frac{2\pi(f_2 - f_1)t}{2}) \cdot \mathrm{sinc}(\frac{2\pi(f_2 + f_1)t}{2}),$$

where $\mathrm{sinc}(x) = \sin(x)/x$ and $\Pi(x)$ denotes the rectangular pulse defined in class, i.e.,

$$\Pi(x) = \begin{cases} 1 & \text{for } |x| < \frac{1}{2} \\ 0 & \text{else.} \end{cases}$$

(a) Show that the inverse Fourier transform of $\Pi(f/f_0)$ is given by $f_0 \cdot \mathrm{sinc}(\pi f_0 t)$.

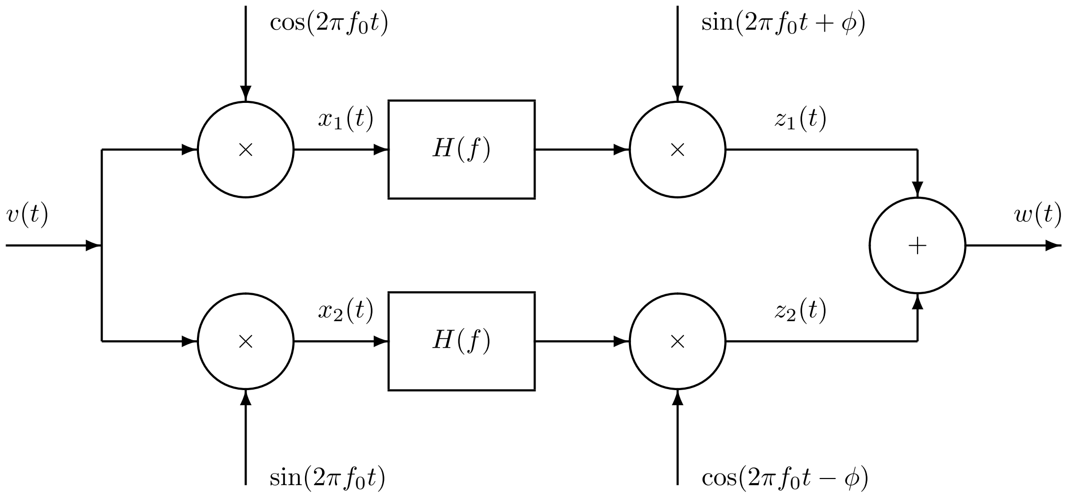
(b) Compute the Fourier transform of $s_1(t)$.

(c) Using the convolution rule, find the Fourier transform $S_2(f)$ of $s_2(t)$. Plot the magnitude of $S_2(f)$.

(d) Is it possible to completely reconstruct either of the two signals $s_1(t)$ and $s_2(t)$ from samples taken at rate $1/T_s$? Justify your answer for each signal separately.

(e) If you answered "yes" for either or both of the two signals, give the smallest sampling period T_s which allows for perfect reconstruction.

7. Consider the system in the block diagram below.



(a) Determine the Fourier transforms $X_1(f)$ and $X_2(f)$ of the signals $x_1(t)$ and $x_2(t)$ as a function of the Fourier transform $V(f)$ of the input signal $v(t)$.

(b) Find the Fourier transforms of the signals $Z_1(f)$ and $Z_2(f)$ of the signals $z_1(t)$ and $z_2(t)$ as a function $V(f)$ and $H(f)$.

(c) Which value of the angle ϕ allows $W(f)$, the Fourier transform of $w(t)$, to be written as

$$W(f) = G(f) \cdot V(f),$$

where $G(f)$ is not a function of $V(f)$? Express $G(f)$ as a function of $H(f)$ and f_0.

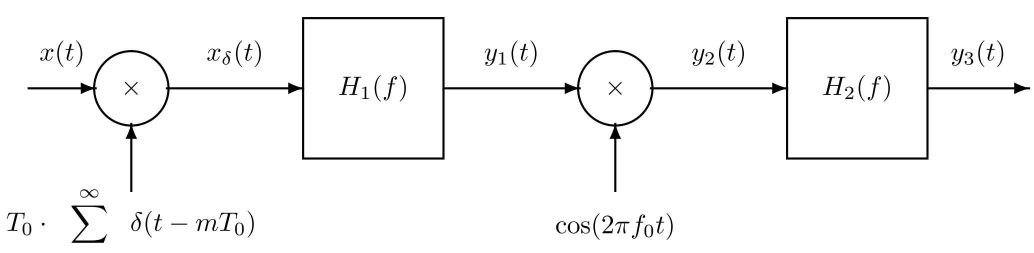
(d) Assume now that $H(f)$ is the transfer function of an ideal lowpass with cut-off frequency $f_c = \frac{f_0}{2}$. Sketch $|G(f)|$ for this case.

Hint: You may need the following identities:

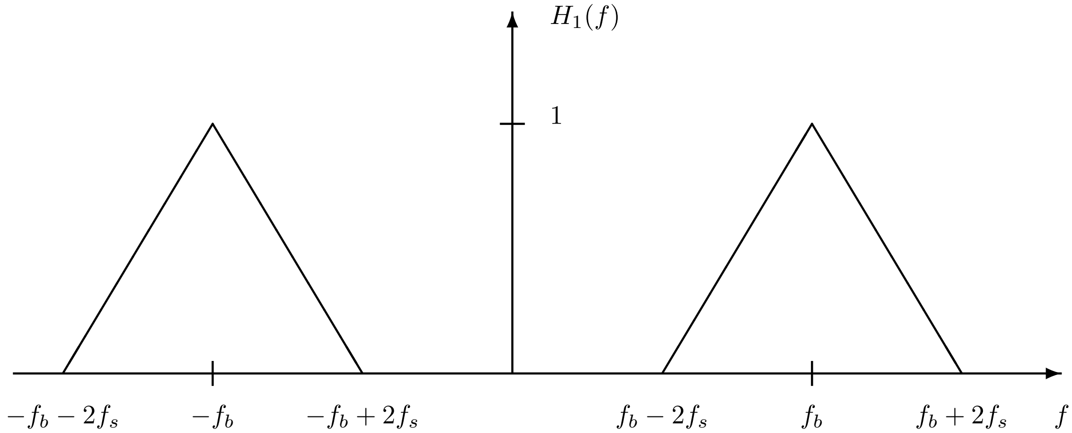
$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

8. **Sampling Systems**
Consider a bandlimited signal $x(t)$ with Fourier transform $X(f)$. ($X(f) = 0$ for $|f| > f_s$.) This signal is input to the following system:



The frequency response $H_1(f)$ of the linear system in the block diagram above is given by:



The sampling rate $f_0 = \frac{1}{T_0}$, the highest signal frequenct f_s, and the center frequency f_b of the band-pass H_1 are related through

$$f_0 = f_b - f_s \qquad \text{and} \qquad f_b \geq 5f_s.$$

(a) Show that the Fourier transform $X_\delta(f)$ of the sampled signal $x_\delta(t)$ is given by

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} X(f - nf_0).$$

Sketch $X_\delta(f)$ for a typical $X(f)$.

(b) Compute the Fourier transform $Y_1(f)$ of the output $y_1(t)$ of the bandpass H_1. Sketch $Y_1(f)$ for a typical $X(f)$.

(c) Compute the Fourier transform $Y_2(f)$ of the signal $y_2(t)$. Sketch $Y_2(f)$ for a typical $X(f)$.

(d) How would you choose the frequency response of the system $H_2(f)$ so that the overall system (between $x(t)$ and $y_3(t)$) does not introduce distortion?

9. **Amplitude Modulation**
Consider the following amplitude modulated signal,

$$x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

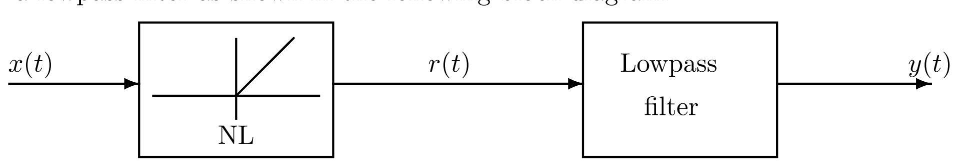
The constant A has been chosen such that $(A + m(t)) > 0$ for all t. Furthermore, the spectrum of the message signal $m(t)$ is bandlimited, such that

$$M(f) = 0 \quad \text{for } |f| > f_m$$

The carrier frequency f_c is much larger than f_m.

(a) For a typical spectrum $M(f)$, sketch the magnitude of the spectrum $X(f)$ of the signal $x(t)$.

(b) The envelope detector discussed in class can be modeled as a nonlinear device followed by a lowpass filter as shown in the following block diagram:



The nonlinear device NL is described by the relationship between its input $x(t)$ and its output $r(t)$ as

$$r(t) = \mathrm{NL}(x(t)) = \begin{cases} x(t) & \text{if } x(t) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume the signal $g(t) = \cos(2\pi f_c t)$ is passed through the above nonlinearity. Sketch the resulting signal $\mathrm{NL}(g(t))$ and show that this signal can be represented as a Fourier series by

$$\mathrm{NL}(g(t)) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{\sin(\frac{\pi}{2}(n-1))}{n-1} + \frac{\sin(\frac{\pi}{2}(n+1))}{n+1} \right) \mathrm{e}^{-jn2\pi f_c t}.$$

(c) Using the result from part (b), sketch and accurately label the magnitude of the spectrum of the signal $r(t)$.

(d) Assume that the lowpass filter in the demodulator is ideal. How would you choose the cutoff frequency of that filter?

10. **AM Stereo Signals**
The use of *quadrature carrier multiplexing* provides the basis for the generation of AM stereo signals. One particular form of such a signal is described by

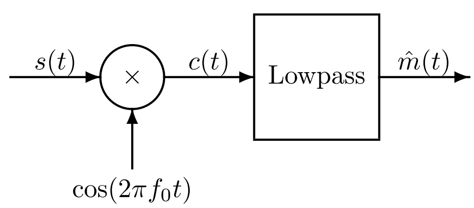
$$s(t) = A_c(m_l(t)\cos(2\pi f_0 t - \frac{\pi}{4}) + m_r(t)\cos(2\pi f_0 t + \frac{\pi}{4})),$$

where f_c is the carrier frequency and $m_l(t)$ and $m_r(t)$ are the left-hand and right-hand output signals of the loudspeakers, respectively. You may assume throughout that $m_l(t)$ and $m_r(t)$ are lowpass signals with bandwidth f_m much smaller than f_0.

(a) Draw a block diagram of a system with inputs $m_l(t)$ and $m_r(t)$ that generates the signal $s(t)$.

(b) Compute the Fourier transform of the signal $s(t)$.

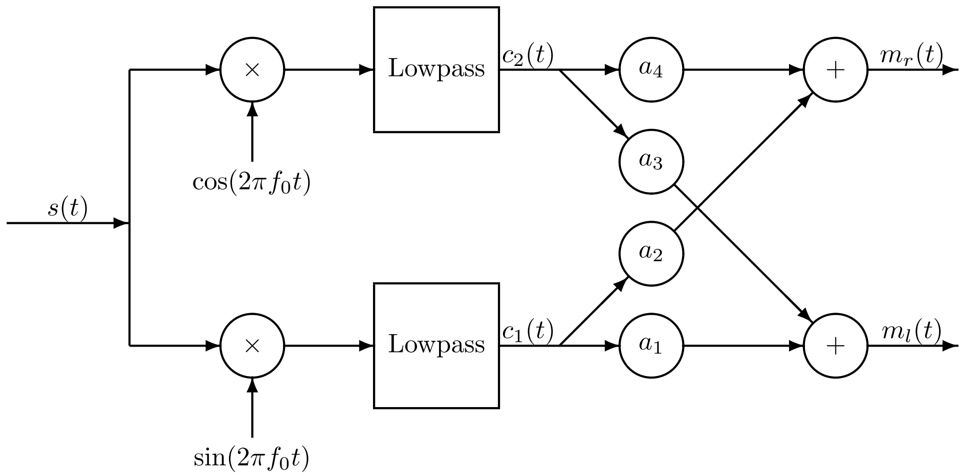
(c) A coherent mono receiver can be implemented as in the following block diagram:



Derive an expression for the signal $c(t)$ and then compute the Fourier transform of the signal $c(t)$.

(d) Assume the lowpass filter to be ideal with cut-off frequency f_c satisfying $f_m < f_c \ll f_0$, find an expression for the output signal $\hat{m}(t)$. Explain how the mono receiver processes the received signal containing stereo signals.

(e) Now a stereo receiver as shown in the following block diagram is employed:



Proceeding as in parts (c) and (d), compute expressions for the two signals $c_1(t)$ and $c_2(t)$. Then determine how the coefficients a_i, $i = 1, 2, 3, 4$ must be chosen such that the original left-hand and right-hand signals $m_l(t)$ and $m_r(t)$ are recovered.

Hint:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$
$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$
$$\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$$