Introduction

- We have focused on the problem of deciding which of *two* possible signals has been transmitted.
  - Binary Signal Sets
- We will generalize the design of optimum (MPE) receivers to signal sets with M signals.
  - M-ary signal sets.
- With binary signal sets *one* bit can be transmitted in each signal period T.
- With M-ary signal sets, $\log_2(M)$ bits are transmitted simultaneously per T seconds.
  - **Example (**$M = 4$**):**

$$\begin{array}{ll} 00 \to s_0(t) & 01 \to s_1(t) \\ 10 \to s_2(t) & 11 \to s_3(t) \end{array}$$

M-ary Hypothesis Testing Problem

- We can formulate the optimum receiver design problem as a hypothesis testing problem:

$$
\begin{aligned}
H_0 \colon R_t &= s_0(t) + N_t \\
H_1 \colon R_t &= s_1(t) + N_t \\
&\vdots \\
H_{M-1} \colon R_t &= s_{M-1}(t) + N_t
\end{aligned}
$$

  with a priori probabilities $\pi_i = \Pr\{H_i\}$, $i = 0, 1, \ldots, M-1$.
- Note:
  - With more than two hypotheses, it is no longer helpful to consider the (likelihood) ratio of pdfs.
  - Instead, we focus on the hypothesis with the *maximum a posteriori (MAP)* probability or the *maximum likelihood (ML)*.

AWGN Channels

- Of most interest in communications are channels where N_t is a white Gaussian noise process.
  - Spectral height $\frac{N_0}{2}$.
- For these channels, the optimum receivers can be found by arguments completely analogous to those for the binary case.
  - Note that with M-ary signal sets, the subspace containing all signals will have up to M dimensions.
- We will determine the optimum receivers by generalizing the optimum binary receivers for AWGN channels.

Starting Point: Binary MPE Decision Rule

- We have shown, that the binary MPE decision rule can be expressed equivalently as
  - either

$$\langle R_t, (s_1(t) - s_0(t)) \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2} \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|s_1(t)\|^2 - \|s_0(t)\|^2}{2}$$

  - or

$$\|R_t - s_0(t)\|^2 - N_0 \ln(\pi_0) \underset{H_0}{\overset{H_1}{\gtrless}} \|R_t - s_1(t)\|^2 - N_0 \ln(\pi_1)$$

- The first expression is most useful for deriving the structure of the optimum receiver.
- The second form is helpful for interpreting the decision rule in signal space.

M-ary MPE Receiver

- The decision rule

$$\langle R_t, (s_1(t) - s_0(t))\rangle \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2} \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|s_1(t)\|^2 - \|s_0(t)\|^2}{2}$$

can be rewritten as

$$Z_1 = \langle R_t, s_1(t)\rangle + \overbrace{\frac{N_0}{2}\ln(\pi_1) - \frac{\|s_1(t)\|^2}{2}}^{\gamma_1} \underset{H_0}{\overset{H_1}{\gtrless}} \langle R_t, s_0(t)\rangle + \underbrace{\frac{N_0}{2}\ln(\pi_0) - \frac{\|s_0(t)\|^2}{2}}_{\gamma_0} = Z_0$$

M-ary MPE Receiver

- The decision rule is easily generalized to M signals:

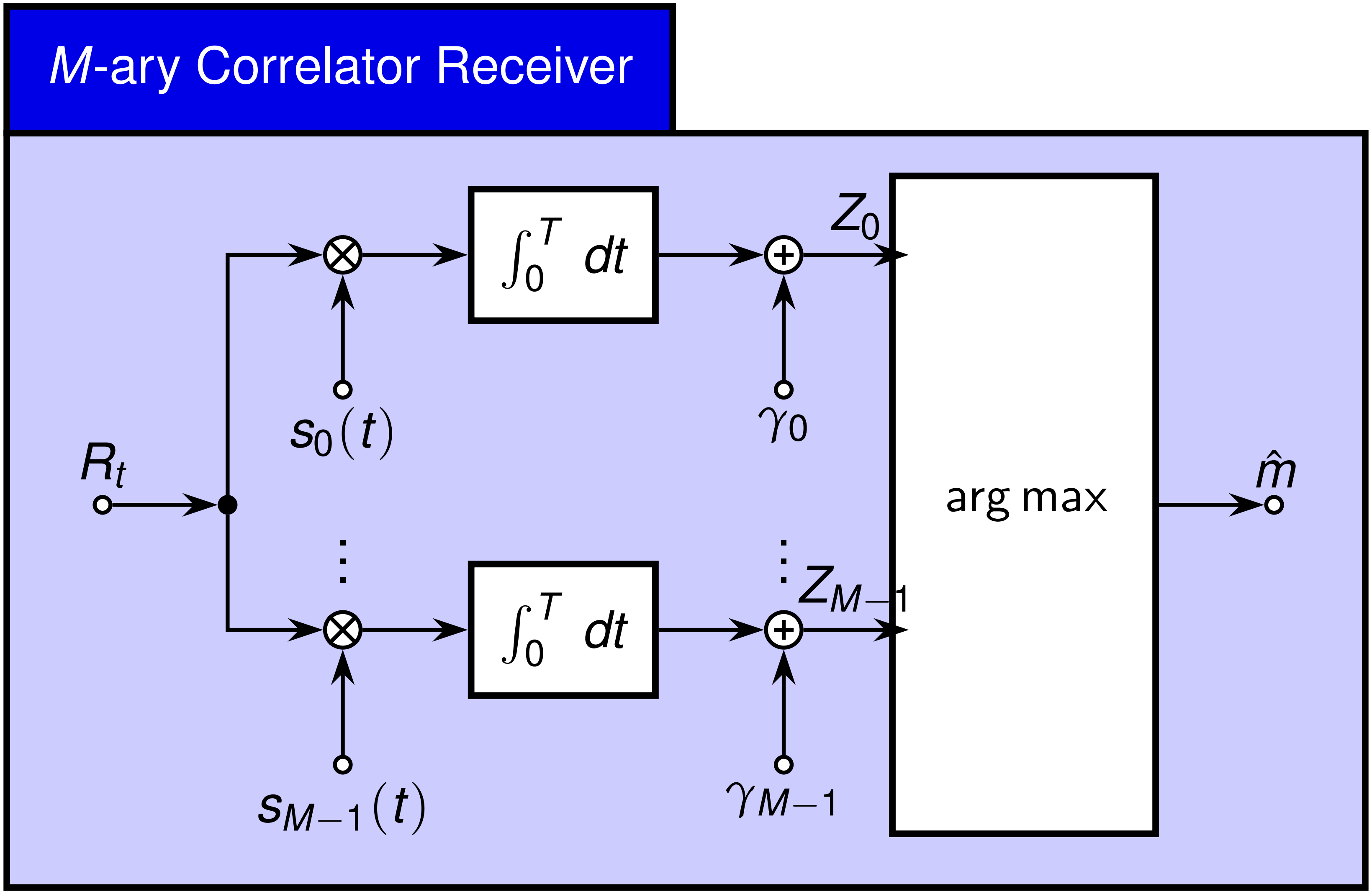
$$\hat{m} = \arg \max_{n=0,\ldots,M-1} \overbrace{\langle R_t, s_n(t) \rangle + \underbrace{\frac{N_0}{2} \ln(\pi_n) - \frac{\|s_n(t)\|^2}{2}}_{\gamma_n}}^{Z_n}$$

- The optimum detector selects the hypothesis with the largest decision statistic Z_n.

M-ary MPE Receiver

- The *bias terms* γ_n account for unequal priors and for differences in signal energy $E_n = \|s_n(t)\|^2$.
- Common terms can be omitted
  - For equally likely signals,
$$\gamma_n = -\frac{\|s_n(t)\|^2}{2}.$$
  - For equal energy signals,
$$\gamma_n = \frac{N_0}{2} \ln(\pi_n)$$
  - For equally likely, equal energy signal,
$$\gamma_n = 0$$

M-ary MPE Receiver



Decision Statistics

- The optimum receiver computes the decision statistics

$$Z_n = \langle R_t, s_n(t) \rangle + \frac{N_0}{2} \ln(\pi_n) - \frac{\|s_n(t)\|^2}{2}.$$

- Conditioned on the m-th signal having been transmitted,
  - All Z_n are Gaussian random variables.
  - Expected value:

$$\mathbf{E}[Z_n|H_m] = \langle s_m(t), s_n(t) \rangle + \frac{N_0}{2} \ln(\pi_n) - \frac{\|s_n(t)\|^2}{2}$$

  - (Co)Variance:

$$\mathbf{E}[Z_j Z_k|H_m] - \mathbf{E}[Z_j|H_m]\mathbf{E}[Z_k|H_m] = \langle s_j(t), s_k(t) \rangle \frac{N_0}{2}$$

Exercise: QPSK Receiver

- Find the optimum receiver for the following signal set with $M = 4$ signals:

$$s_n(t) = \sqrt{\frac{2E}{T}} \cos(2\pi t/T + n\pi/2) \quad \text{for } 0 \leq t \leq T \text{ and } n = 0, \ldots,$$

Decision Regions

- The decision regions Γ_n and error probabilities are best understood by generalizing the binary decision rule:

$$\|R_t - s_0(t)\|^2 - N_0 \ln(\pi_0) \underset{H_0}{\overset{H_1}{\gtrless}} \|R_t - s_1(t)\|^2 - N_0 \ln(\pi_1)$$

- For M-ary signal sets, the decision rule generalizes to

$$\hat{m} = \arg \min_{n=0,\ldots,M-1} \|R_t - s_n(t)\|^2 - N_0 \ln(\pi_n).$$

  - This simplifies to

$$\hat{m} = \arg \min_{n=0,\ldots,M-1} \|R_t - s_n(t)\|^2$$

    for equally likely signals.
  - The optimum receiver decides in favor of the signal $s_n(t)$ that is *closest* to the received signal.

Decision Regions (equally likely signals)

- For discussing decision regions, it is best to express the decision rule in terms of the representation obtained with the orthonormal basis $\{\Phi_k\}$, where
  - basis signals Φ_k span the space that contains all signals $s_n(t)$, with $n = 0, \ldots, M-1$.
    - Recall that we can obtain these basis signals via the Gram-Schmidt procedure from the signal set.
    - There are at most M orthonormal bases.
- Because of Parseval's relationship, an equivalent decision rule is

$$\hat{m} = \arg \min_{n=0,\ldots,M-1} \|\vec{R} - \vec{s}_n\|^2,$$

where $\vec{R}$ has elements $R_k = \langle R_t, \Phi_k(t) \rangle$ and $\vec{s}_n$ has element $s_{n,k} = \langle s_n(t), \Phi_k(t) \rangle$.

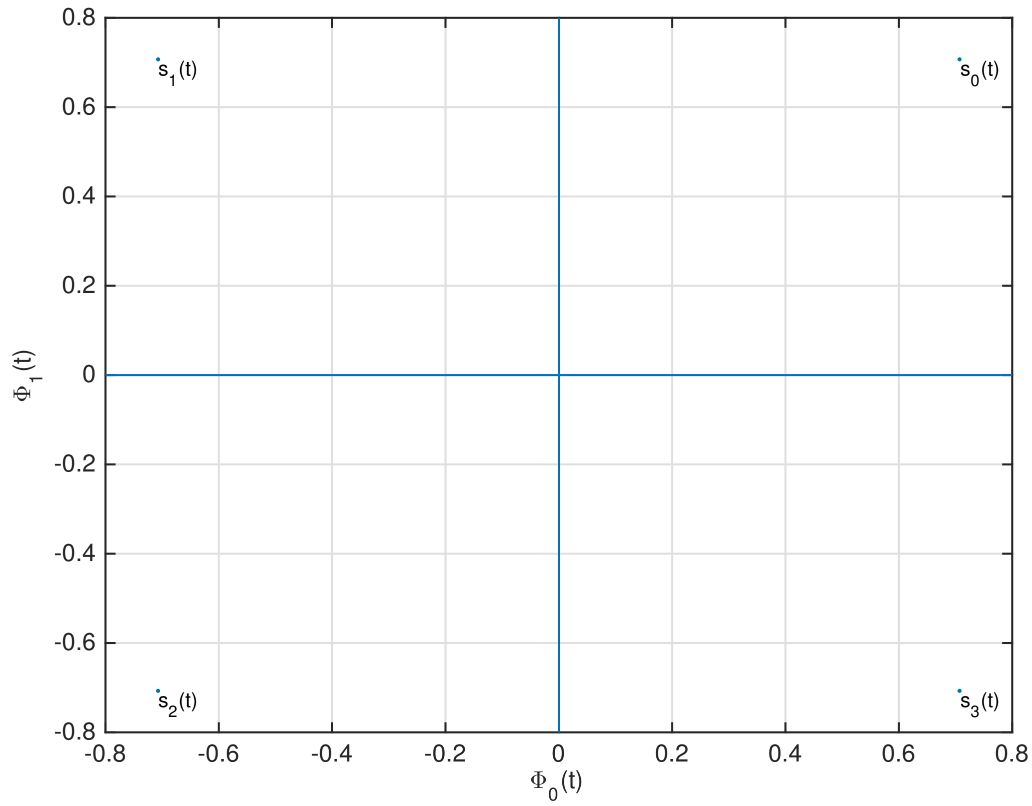
Decision Regions

- The decision region Γ_n where the detector decides that the n-th signal was sent is

$$\Gamma_n = \{\vec{r} : \|\vec{r} - \vec{s}_n\| < \|\vec{r} - \vec{s}_m\| \text{for all } m \neq n\}.$$

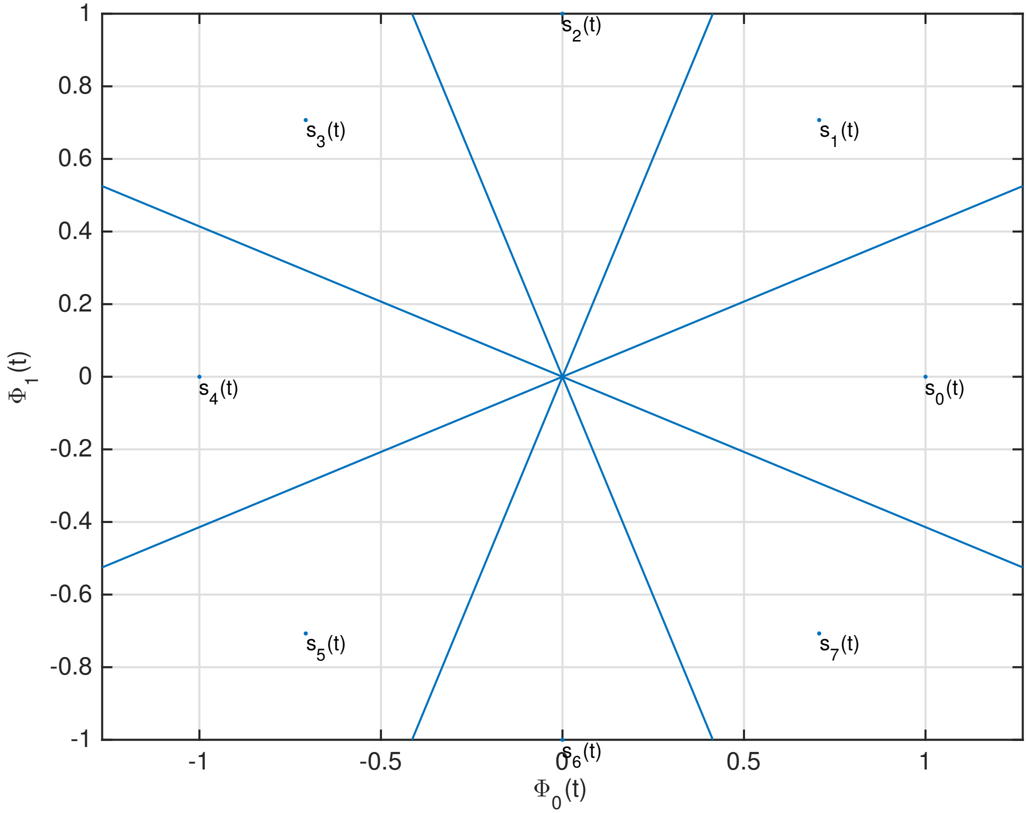
  - The decision region Γ_n is the set of all points $\vec{r}$ that are closer to $\vec{s}_n$ than to any other signal point.
- The decision regions are formed by linear segments that are *perpendicular bisectors* between pairs of signal points.
  - The resulting partition is also called a *Voronoi partition.*

Example: QPSK



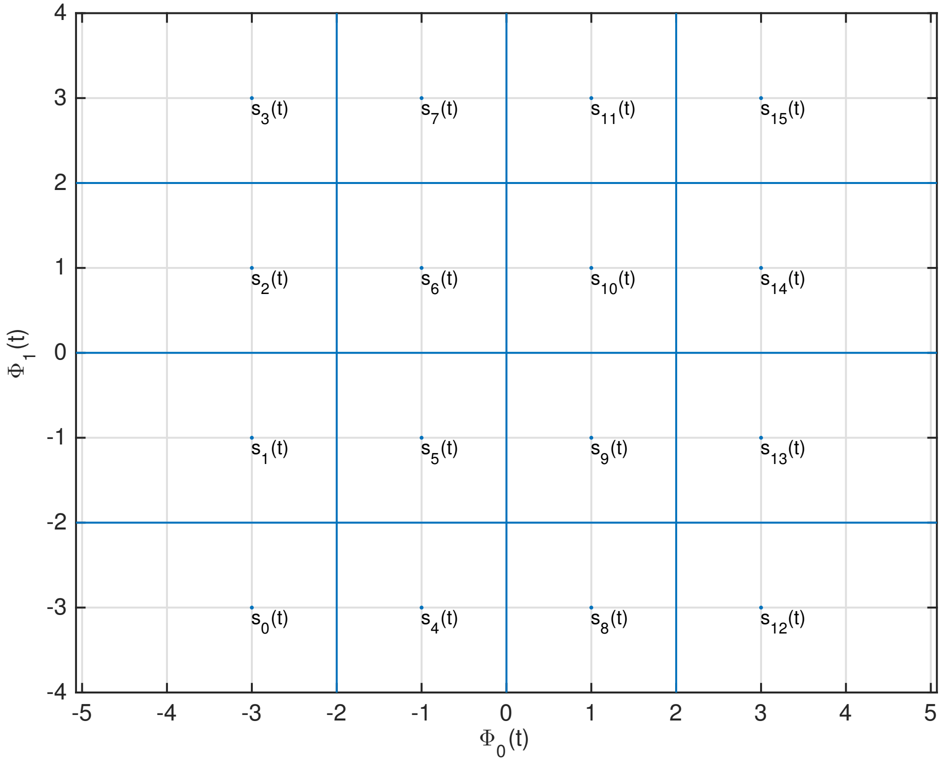
$$s_n(t) = \sqrt{2/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3.$$

Example: 8-PSK



$$s_n(t) = \sqrt{2/T}\cos(2\pi f_c t + n \cdot \pi/4), \text{ for } n = 0, \ldots, 7.$$

Example: 16-QAM



$$s_n(t) = \sqrt{2/T}(A_I \cdot \cos(2\pi f_c t) + A_Q \cdot \sin(2\pi f_c t))$$

$$\text{with } A_I, A_Q \in \{-3, -1, 1, 3\}.$$

Symbol Energy and Bit Energy

- We have seen that error probabilities decrease when the signal energy increases.
  - Because the distance between signals increase.
- We will see further that error rates in AWGN channels depend only on
  - the signal-to-noise ratio $\frac{E_b}{N_0}$, where E_b is the average energy per bit, and
  - the geometry of the signal constellation.
- To focus on the impact of the signal geometry, we will fix either
  - the average energy per symbol $E_s = \frac{1}{M} \sum_{n=0}^{M-1} \|s_n(t)\|^2$ or
  - the average energy per bit $E_b = \frac{E_s}{\log_2(M)}$

Example: QPSK

- QPSK signals are given by

$$s_n(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3.$$

- Each of the four signals $s_n(t)$ has energy

$$E_n = \|s_n(t)\|^2 = E_s.$$

- Hence,
  - the average symbol energy is E_s
  - the average bit energy is $E_b = \frac{E_s}{\log_2(4)} = \frac{E_s}{2}$

Example: 8-PSK

- 8-PSK signals are given by

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/4), \text{ for } n = 0, \ldots, 7.$$

- Each of the eight signals $s_n(t)$ has energy

$$E_n = \|s_n(t)\|^2 = E_s.$$

- Hence,
  - the average symbol energy is E_s
  - the average bit energy is $E_b = \frac{E_s}{\log_2(8)} = \frac{E_s}{3}$

Example: 16-QAM

- 16-QAM signals can be written as

$$s_n(t) = \sqrt{\frac{2E_0}{T}} \left(a_I \cdot \cos(2\pi f_c t) + a_Q \cdot \sin(2\pi f_c t) \right)$$

  with $a_I, a_Q \in \{-3, -1, 1, 3\}$.
- There are
  - 4 signals with energy $(1^2 + 1^2)E_0 = 2E_0$
  - 8 signals with energy $(3^2 + 1^2)E_0 = 10E_0$
  - 4 signals with energy $((3^2 + 3^2)E_0 = 18E_0$
- Hence,
  - the average symbol energy is $10E_0$
  - the average bit energy is $E_b = \frac{E_s}{\log_2(16)} = \frac{5E_0}{2}$

Energy Efficiency

- We will see that the influence of the signal geometry is captured by the energy efficiency

$$\eta_P = \frac{d_{\text{min}}^2}{E_b}$$

where d_{min} is the smallest distance between any pair of signals in the constellation.

- Examples:
  - **QPSK:** $d_{\text{min}} = \sqrt{2E_s}$ and $E_b = \frac{E_s}{2}$, thus $\eta_P = 4$.
  - **8-PSK:** $d_{\text{min}} = \sqrt{(2-\sqrt{2})E_s}$ and $E_b = \frac{E_s}{3}$, thus $\eta_P = 3 \cdot (2-\sqrt{2}) \approx 1.75$.
  - **16-QAM:** $d_{\text{min}} = \sqrt{2E_0}$ and $E_b = \frac{5E_0}{2}$, thus $\eta_P = \frac{8}{5}$.
- Note that energy efficiency decreases with the size of the constellation for 2-dimensional constellations.

Computing Probability of Symbol Error

- When decision boundaries intersect at right angles, then it is possible to compute the error probability exactly in closed form.
  - The result will be in terms of the *Q*-function.
  - This happens whenever the signal points form a rectangular grid in signal space.
  - Examples: QPSK and 16-QAM
- When decision regions are not rectangular, then closed form expressions are not available.
  - Computation requires integrals over the *Q*-function.
  - We will derive good bounds on the error rate for these cases.
  - For exact results, numerical integration is required.

Illustration: 2-dimensional Rectangle

- Assume that the n-th signal was transmitted and that the representation for this signal is $\vec{s}_n = (s_{n,0}, s_{n,1})'$.
- Assume that the decision region Γ_n is a rectangle

$$\Gamma_n = \{\vec{r} = (r_0, r_1)' : s_{n,0} - a_1 < r_0 < s_{n,0} + a_2 \text{ and } s_{n,1} - b_1 < r_1 < s_{n,1} + b_2\}.$$

  - Note: we have assumed that the sides of the rectangle are parallel to the axes in signal space.
  - Since *rotation* and *translation* of signal space do not affect distances this can be done without affecting the error probability.
- **Question:** What is the conditional error probability, assuming that $s_n(t)$ was sent.

Illustration: 2-dimensional Rectangle

- In terms of the random variables $R_k = \langle R_t, \Phi_k \rangle$, with $k = 0, 1$, an error occurs if

$$\overbrace{(R_0 \leq s_{n,0} - a_1 \text{ or } R_0 \geq s_{n,0} + a_2)}^{\text{error event 1}} \text{ or}$$
$$\underbrace{(R_1 \leq s_{n,1} - b_1 \text{ or } R_1 \geq s_{n,1} + b_2)}_{\text{error event 2}}.$$

- Note that the two error events are not mutually exclusive.
- Therefore, it is better to consider correct decisions instead, i.e., $\vec{R} \in \Gamma_n$:

$$s_{n,0} - a_1 < R_0 < s_{n,0} + a_2 \textbf{ and } s_{n,1} - b_1 < R_1 < s_{n,1} + b_2$$

Illustration: 2-dimensional Rectangle

- We know that R_0 and R_1 are
  - independent - because Φ_k are orthogonal
  - with means $s_{n,0}$ and $s_{n,1}$, respectively
  - variance $\frac{N_0}{2}$.
- Hence, the probability of a correct decision is

$$
\begin{aligned}
\Pr\{c|s_n\} =& \Pr\{-a_1 < N_0 < a_2\} \cdot \Pr\{-b_1 < N_1 < b_2\} \\
=& \int_{-a_1}^{a_2} p_{R_0|s_n}(r_0)\, dr_0 \cdot \int_{-b_1}^{b_2} p_{R_1|s_n}(r_1)\, dr_1 \\
=& (1 - Q\left(\frac{a_1}{\sqrt{N_0/2}}\right) - Q\left(\frac{a_2}{\sqrt{N_0/2}}\right)) \cdot \\
& (1 - Q\left(\frac{b_1}{\sqrt{N_0/2}}\right) - Q\left(\frac{b_2}{\sqrt{N_0/2}}\right)).
\end{aligned}
$$

Exercise: QPSK

- Find the error rate for the signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3.$$

- **Answer:** (Recall $\eta_P = \frac{d^2_{\min}}{E_b} = 4$ for QPSK)

$$\begin{aligned}
\Pr\{e\} &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \\
&= 2Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$

Exercise: 16-QAM

(Recall $\eta_P = \frac{d^2_{\text{min}}}{E_b} = \frac{8}{5}$ for 16-QAM)

- Find the error rate for the signal set $(a_I, a_Q \in \{-3, -1, 1, 3\})$

$$s_n(t) = \sqrt{2E_0/T} a_I \cdot \cos(2\pi f_c t) + \sqrt{2E_0/T} a_Q \cdot \sin(2\pi f_c t)$$

- **Answer:** $(\eta_P = \frac{d^2_{\text{min}}}{E_b} = 4)$

$$\begin{aligned}
\Pr\{e\} &= 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{2E_0}{N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$