Example: Gaussian Hypothesis Testing

- The most important hypothesis testing problem for communications over AWGN channels is

$$H_0: \vec{R} \sim N(\vec{m}_0, \sigma^2 I)$$
$$H_1: \vec{R} \sim N(\vec{m}_1, \sigma^2 I)$$

- This problem arises when
  - one of two known signals is transmitted over an AWGN channel, and
  - a linear analog frontend is used.
- Note that
  - the conditional means are different — reflecting different signals
  - covariance matrices are the same — since they depend on noise only.
  - components of $\vec{R}$ are independent — indicating that the frontend projects R_t onto orthogonal bases.

Resulting Log-Likelihood Ratio

- For this problem, the log-likelihood ratio simplifies to

$$
\begin{aligned}
L(\vec{R}) &= \frac{1}{2\sigma^2}\sum_{k=1}^{n}(R_k - m_{0k})^2 - (R_k - m_{1k})^2 \\
&= \frac{1}{2\sigma^2}(\|\vec{R} - \vec{m}_0\|^2 - \|\vec{R} - \vec{m}_1\|^2) \\
&= \frac{1}{2\sigma^2}\left(2\langle \vec{R}, \vec{m}_1 - \vec{m}_0\rangle - (\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2)\right)
\end{aligned}
$$

  - The second expressions shows that the *Euclidean distance* between observations $\vec{R}$ and means $\vec{m}_i$ plays a central role in Gaussian hypothesis testing.
  - The last expression highlights the projection of the observation $\vec{R}$ onto the difference between the means $\vec{m}_i$.

MPE Decision Rule

- With the above log-liklihood ratio, the MPE decision rule becomes equivalently
  - either

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}$$

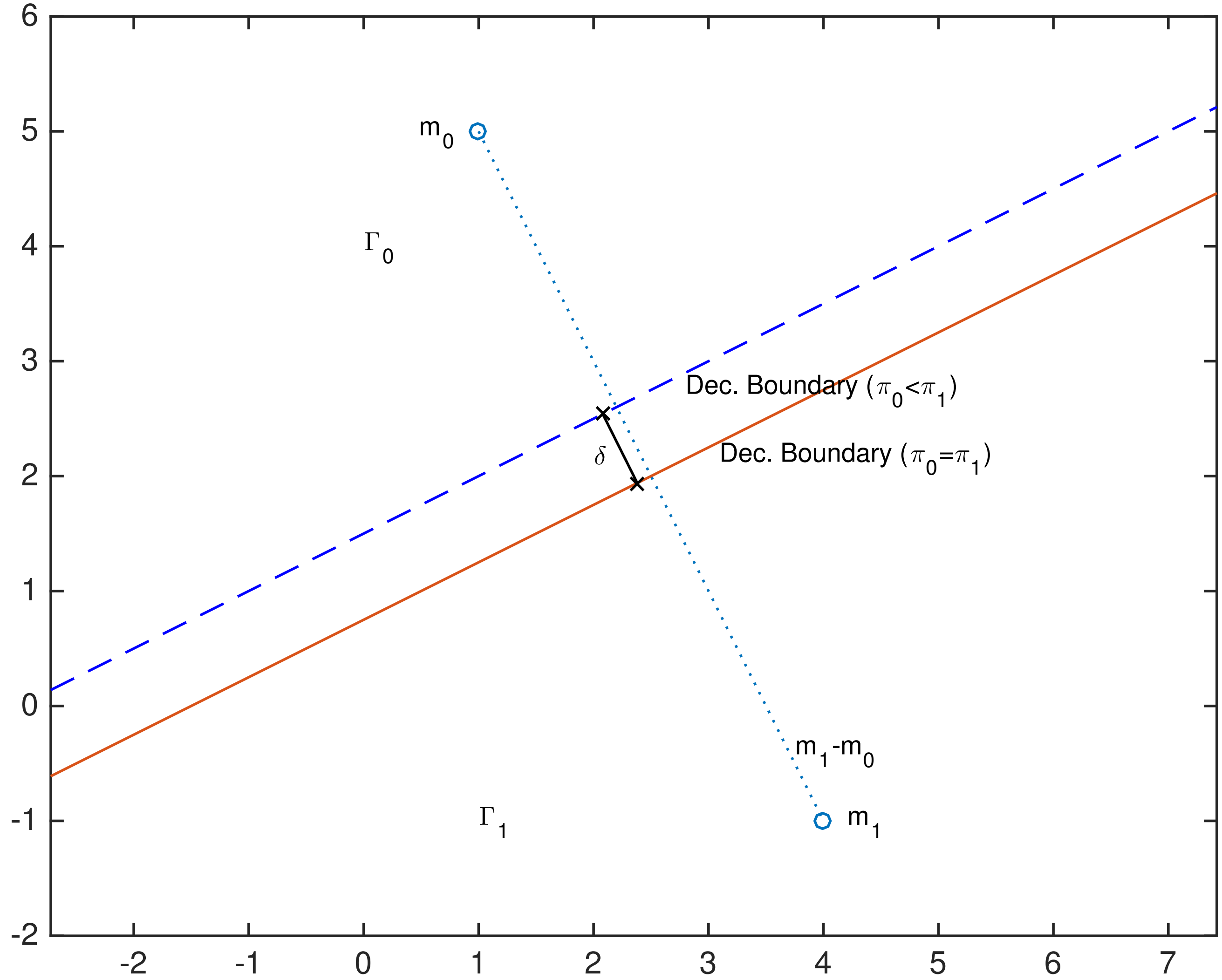
  - or

$$\|\vec{R} - \vec{m}_0\|^2 - 2\sigma^2 \ln(\pi_0) \underset{H_0}{\overset{H_1}{\gtrless}} \|\vec{R} - \vec{m}_1\|^2 - 2\sigma^2 \ln(\pi_1)$$

Decision Regions

- The MPE decision rule divides $\mathbb{R}^n$ into two half planes that are the decision regions Γ_0 and Γ_1.
- The dividing line (decision boundary) between the regions is *perpendicular to* $\vec{m}_1 - \vec{m}_0$.
  - This is a consequence of the inner product in the first form of the decision rule.
- If the priors π_0 and π_1 are equal, then the decision boundary passes through the midpoint $\frac{\vec{m}_0+\vec{m}_1}{2}$.
  - For unequal priors, the decision boundary is shifted towards the mean of the *less likely* hypothesis.
  - The distance of this shift equals $\delta = \frac{2\sigma^2 |\ln(\pi_0/\pi_1)|}{\|\vec{m}_1 - \vec{m}_0\|}$.
  - This follows from the (squared) distances in the second form of the decision rule.

Decision Regions



Probability of Error

- **Question:** What is the probability of error with the MPE decision rule?
  - Using MPE decision rule

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}$$

- **Plan:**
  - Find conditional densities of $\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle$ under H_0 and H_1.
  - Find conditional error probabilities

$$\int_{\Gamma_i} p_{\vec{R}|H_j}(\vec{r}|H_j)\, d\vec{r} \text{ for } i \neq j.$$

  - Find average probability of error.

Conditional Distributions

- Since $\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle$ is a linear transformation and $\vec{R}$ is Gaussian, the conditional distributions are Gaussian.

$$H_0\colon N(\underbrace{\langle \vec{m}_0, \vec{m}_1 \rangle - \|\vec{m}_0\|^2}_{\mu_0}, \underbrace{\sigma^2 \|\vec{m}_0 - \vec{m}_1\|^2}_{\sigma_m^2})$$

$$H_1\colon N(\underbrace{\|\vec{m}_1\|^2 - \langle \vec{m}_0, \vec{m}_1 \rangle}_{\mu_1}, \underbrace{\sigma^2 \|\vec{m}_0 - \vec{m}_1\|^2}_{\sigma_m^2})$$

Conditional Error Probabilities

- The MPE decision rule compares

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}}_{\gamma}$$

- Resulting conditional probabilities of error

$$\Pr\{e|H_0\} = Q\left(\frac{\gamma - \mu_0}{\sigma_m}\right) = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} + \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right)$$

$$\Pr\{e|H_1\} = Q\left(\frac{\mu_1 - \gamma}{\sigma_m}\right) = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} - \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right)$$

Average Probability of Error

- The average error probability equals

$$
\begin{aligned}
\Pr\{e\} &= \Pr\{\text{decide } H_0 | H_1\} \Pr\{H_1\} + \Pr\{\text{decide } H_1 | H_0\} \Pr\{H_0\} \\
&= \pi_0 Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} + \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|} \right) + \\
&\quad \pi_1 Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} - \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|} \right)
\end{aligned}
$$

- Important special case: $\pi_0 = \pi_1 = \frac{1}{2}$

$$
\Pr\{e\} = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} \right)
$$

  - The error probability depends on the ratio of
    - distance between means $\|\vec{m}_0 - \vec{m}_1\|$
    - and noise standard deviation

Maximum-Likelihood (ML) Decision Rule

- The maximum-likelihood decision rule disregards priors and decides for the hypothesis with higher likelihood.
- **ML Decision rule:**

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

or equivalently, in terms of the log-likelihood,

$$L(\vec{R}) = \ln\left(\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

- Obviously, the ML decision is equivalent to the MPE rule when the priors are equal.
- In the Gaussian case, the ML rule does not require knowledge of the noise variance.

A-Posteriori Probability

- By Bayes rule, the probability of hypothesis H_i *after* observing $\vec{R}$ is

$$\Pr\{H_i|\vec{R}=\vec{r}\} = \frac{\pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)}{p_{\vec{R}}(\vec{r})},$$

  where $p_{\vec{R}}(\vec{r})$ is the unconditional pdf of $\vec{R}$

$$p_{\vec{R}}(\vec{r}) = \sum_i \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i).$$

- **Maximum A-Posteriori (MAP) decision rule:**

$$\Pr\{H_1|\vec{R}=\vec{r}\} \underset{H_0}{\overset{H_1}{\gtrless}} \Pr\{H_0|\vec{R}=\vec{r}\}$$

  - **Interpretation:** Decide in favor of the hypothesis that is more likely given the observed signal $\vec{R}$.

The MAP and MPE Rules are Equivalent

- The MAP and MPE rules are equivalent: the MAP decision rule achieves the minimum probability of error.
- The MAP rule can be written as

$$\frac{\Pr\{H_1|\vec{R}=\vec{r}\}}{\Pr\{H_0|\vec{R}=\vec{r}\}} \underset{H_0}{\overset{H_1}{\gtrless}} 1.$$

- Inserting $\Pr\{H_i|\vec{R}=\vec{r}\} = \frac{\pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)}{p_{\vec{R}}(\vec{r})}$ yields

$$\frac{\pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)}{\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

  - This is obviously equal to the MPE rule

$$\frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}.$$

More than Two Hypotheses

- Frequently, more than two hypotheses must be considered:

$$
\begin{aligned}
H_0 &: \vec{R} \sim p_{\vec{R}|H_0}(\vec{r}|H_0) \\
H_1 &: \vec{R} \sim p_{\vec{R}|H_1}(\vec{r}|H_1) \\
&\vdots \\
H_M &: \vec{R} \sim p_{\vec{R}|H_M}(\vec{r}|H_M)
\end{aligned}
$$

- In these cases, it is no longer possible to reduce the decision rules to
  - the computation of the likelihood ratio
  - followed by comparison to a threshold

More than Two Hypotheses

- Instead the decision rules take the following forms
  - **MPE rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)$$

  - **ML rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} p_{\vec{R}|H_i}(\vec{r}|H_i)$$

  - **MAP rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} \Pr\{H_i|\vec{R} = \vec{r}\}$$

More than Two Hypotheses: The Gaussian Case

- When the hypotheses are of the form H_i: $\vec{R} \sim N(\vec{m}_i, \sigma^2 I)$, then the decision rules become:
  - **MPE and MAP decision rules:**

$$\hat{m} = \arg \min_{i \in \{0,\ldots,M-1\}} \|\vec{r} - \vec{m}_i\|^2 - 2\sigma^2 \ln(\pi_i)$$

$$= \arg \max_{i \in \{0,\ldots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle + \sigma^2 \ln(\pi_i) - \frac{\|\vec{m}_i\|^2}{2}$$

  - **ML decision rule:**

$$\hat{m} = \arg \min_{i \in \{0,\ldots,M-1\}} \|\vec{r} - \vec{m}_i\|^2$$

$$= \arg \max_{i \in \{0,\ldots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle - \frac{\|\vec{m}_i\|^2}{2}$$

    - This is also the MPE rule when the priors are all equal.

Take-Aways

- The conditional densities $p_{\vec{R}|H_i}(\vec{r}|H_i)$ play a key role.
- **MPE decision rule:**
  - Binary hypotheses:

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

  - M hypotheses:

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i).$$

Take-Aways

- For the Gaussian case (different means, equal variance), decisions are based on the Euclidean distance between observations $\vec{R}$ and conditional means $\vec{m}_i$:

$$\hat{m} = \arg \min_{i \in \{0,\ldots,M-1\}} \|\vec{r} - \vec{m}_i\|^2 - 2\sigma^2 \ln(\pi_i)$$

$$= \arg \max_{i \in \{0,\ldots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle + \sigma^2 \ln(\pi_i) - \frac{\|\vec{m}_i\|^2}{2}$$