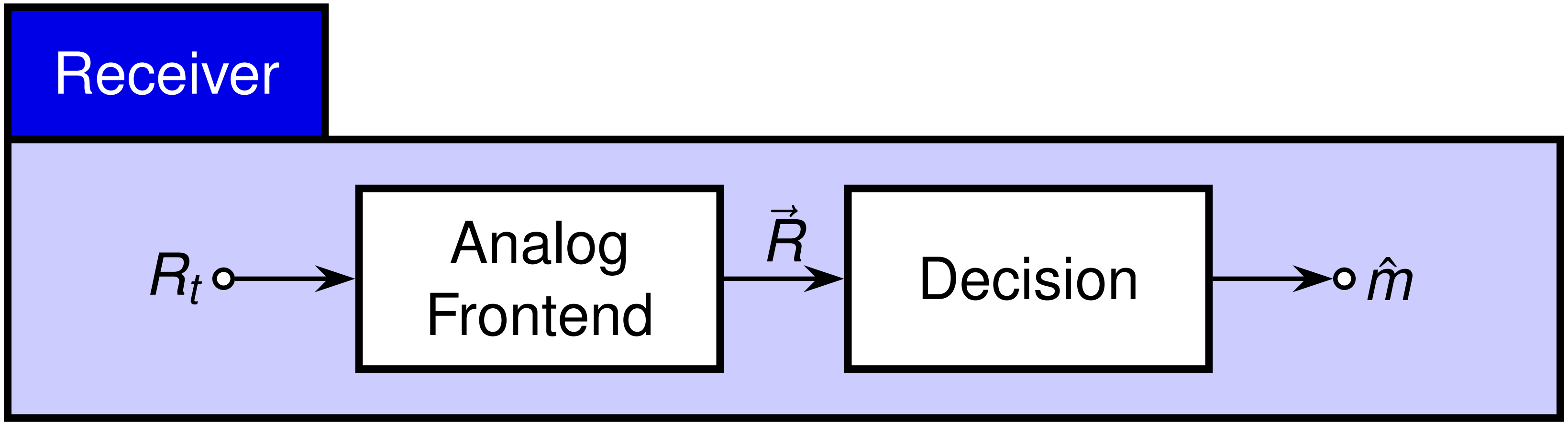
Structure of a Generic Receiver



- Receivers consist of:
  - an *analog frontend*: maps observed signal R_t to decision statistic $\vec{R}$.
  - *decision device*: determines which symbol $\hat{m}$ was sent based on observation of $\vec{R}$.
- Focus on designing optimum frontend.

Problem Formulation and Assumptions

- In terms of the received signal R_t, we can formulate the following decision problem:

$$H_0 \colon R_t = s_0(t) + N_t \text{ for } 0 \leq t \leq T$$
$$H_1 \colon R_t = s_1(t) + N_t \text{ for } 0 \leq t \leq T$$

- **Assumptions:**
  - N_t is whithe Gaussian noise with spectral height $\frac{N_0}{2}$.
  - N_t is independent of the transmitted signal.
- **Objective:** Determine the optimum receiver frontend.

Starting Point: KL-Expansion

- Under the i-th hypothesis, the received signal R_t can be represented over $0 \leq t \leq T$ via the expansion

$$H_i\colon\ R_t = \sum_{j=0}^{\infty} R_j \Phi_j(t) = \sum_{j=0}^{\infty} (s_{ij} + N_j)\Phi_j(t).$$

- **Recall:**
  - If the above representation yields *uncorrelated* coefficients R_j, then this is a Karhunen-Loeve expansion.
  - Since N_t is white, *any* orthonormal basis $\{\Phi_j(t)\}$ yields a Karhunen-Loeve expansion.
- **Insight:**
  - We can *choose* a basis $\{\Phi_j(t)\}$ that produces a low-dimensional representation for all signals $s_i(t)$.

Constructing a Good Basis

- Consider the complete, but not necessarily orthonormal, basis
$$\{s_0(t), s_1(t), \Psi_0(t), \Psi_1(t), \ldots\}.$$
where $\{\Psi_j(t)\}$ is any complete basis over $0 \leq t \leq T$ (e.g., the Fourier basis).
- Then, the Gram-Schmidt procedure is used to convert the above basis into an orthonormal basis $\{\Phi_j\}$.

Properties of Resulting Basis

- **Notice:** with this construction
  - only the first $M \leq 2$ basis functions $\Phi_j(t)$, $j < M \leq 2$ are dependent on the signals $s_i(t)$, $i \leq 2$.
    - I.e., for each $j < M$,
      $$\langle s_i(t), \Phi_j(t) \rangle \neq 0 \text{ for at least one } i = 0, 1$$
    - Recall, $M < 2$ if signals are not linearly independent.
  - The remaining basis functions $\Phi_j(t)$, $j \geq M$ are orthogonal to the signals $s_i(t)$, $i \leq 2$
    - I.e., for each $j \geq M$,
      $$\langle s_i(t), \Phi_j(t) \rangle = 0 \text{ for all } i = 0, 1$$

Back to the Decision Problem

- Our decision problem can now be written in terms of the representation

$$H_0\colon R_t = \underbrace{\sum_{j=0}^{M-1} (s_{0j} + N_j)\Phi_j(t)}_{\text{signal + noise}} + \underbrace{\sum_{j=M}^{\infty} N_j \Phi_j(t)}_{\text{noise only}}$$

$$H_1\colon R_t = \underbrace{\sum_{j=0}^{M-1} (s_{1j} + N_j)\Phi_j(t)}_{\text{signal + noise}} + \underbrace{\sum_{j=M}^{\infty} N_j \Phi_j(t)}_{\text{noise only}}$$

where

$$s_{ij} = \langle s_i(t), \Phi_j(t) \rangle$$
$$N_j = \langle N_t, \Phi_j(t) \rangle$$

  - Note that N_j are independent, Gaussian random variables, $N_j \sim N(0, \frac{N_0}{2})$

Vector Version of Decision Problem

- The received signal R_t and its representation $\vec{R} = \{R_j\}$ are equivalent.
  - Via the basis $\{\Phi_j\}$ one can be obtained from the other.
- Therefore, the decision problem can be written in terms of the representations

$$H_0\colon \vec{R} = \vec{s}_0 + \vec{N}$$
$$H_1\colon \vec{R} = \vec{s}_1 + \vec{N}$$

where

- all vectors are of infinite length,
- the elements of $\vec{N}$ are i.i.d., zero mean Gaussian,
- all elements s_{ij} with $j \geq M$ are zero.

Reducing the Number of Dimensions

- We can write the conditional pdfs for the decision problem

$$H_0\colon \vec{R} \sim \prod_{j=0}^{M-1} p_N(r_j - s_{0j}) \cdot \prod_{j=M}^{\infty} p_N(r_j)$$

$$H_1\colon \vec{R} \sim \prod_{j=0}^{M-1} p_N(r_j - s_{1j}) \cdot \prod_{j=M}^{\infty} p_N(r_j)$$

where $p_N(r)$ denotes a Gaussian pdf with zero mean and variance $\frac{N_0}{2}$.

Reducing the Number of Dimensions

- The optimal decision relies on the likelihood ratio

$$L(\vec{R}) = \frac{\prod_{j=0}^{M-1} p_N(r_j - s_{0j}) \cdot \prod_{j=M}^{\infty} p_N(r_j)}{\prod_{j=0}^{M-1} p_N(r_j - s_{1j}) \cdot \prod_{j=M}^{\infty} p_N(r_j)} = \frac{\prod_{j=0}^{M-1} p_N(r_j - s_{0j})}{\prod_{j=0}^{M-1} p_N(r_j - s_{1j})}$$

- The likelihood ratio depends only on the first M dimensions of $\vec{R}$!
  - Dimensions greater than or equal to M are *irrelevant* for the decision problem.
  - Only the the first M dimension need to be computed for optimal decisions.

Reduced Decision Problem

- The following decision problem with M dimensions is equivalent to our original decision problem (assumes $M = 2$):

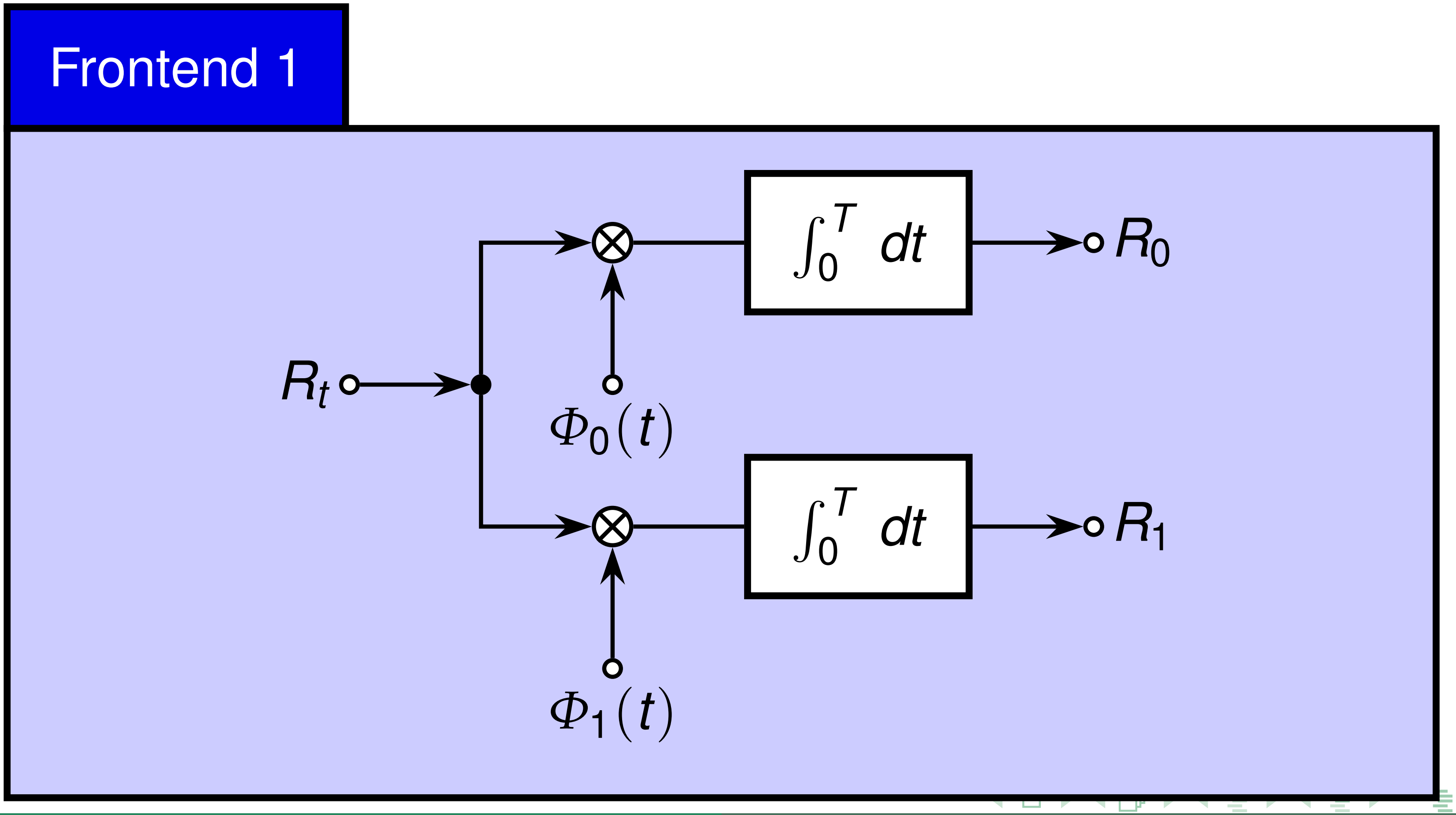
$$H_0\colon \vec{R} = \begin{pmatrix} s_{00} \\ s_{01} \end{pmatrix} + \begin{pmatrix} N_0 \\ N_1 \end{pmatrix} = \vec{s}_0 + \vec{N} \sim \mathrm{N}(\vec{s}_0, \frac{N_0}{2} I)$$

$$H_1\colon \vec{R} = \begin{pmatrix} s_{10} \\ s_{11} \end{pmatrix} + \begin{pmatrix} N_0 \\ N_1 \end{pmatrix} = \vec{s}_1 + \vec{N} \sim \mathrm{N}(\vec{s}_1, \frac{N_0}{2} I)$$

- When $s_0(t)$ and $s_1(t)$ are linearly dependent, i.e., $s_1(t) = a \cdot s_0(t)$, then $M = 1$ and the decision problem becomes one-dimensional.

Optimal Frontend - Version 1

- From the above discussion, we can conclude that an optimal frontend is given by.



Optimum Receiver - Version 1

- Note that the optimum frontend projects the received signal R_t into to signal subspace spanned by the signals $s_i(t)$.
  - Recall that the first basis functions $\Phi_j(t)$, $j < M$, are obtained from the signals.
- We know how to solve the resulting, M-dimensional decision problem

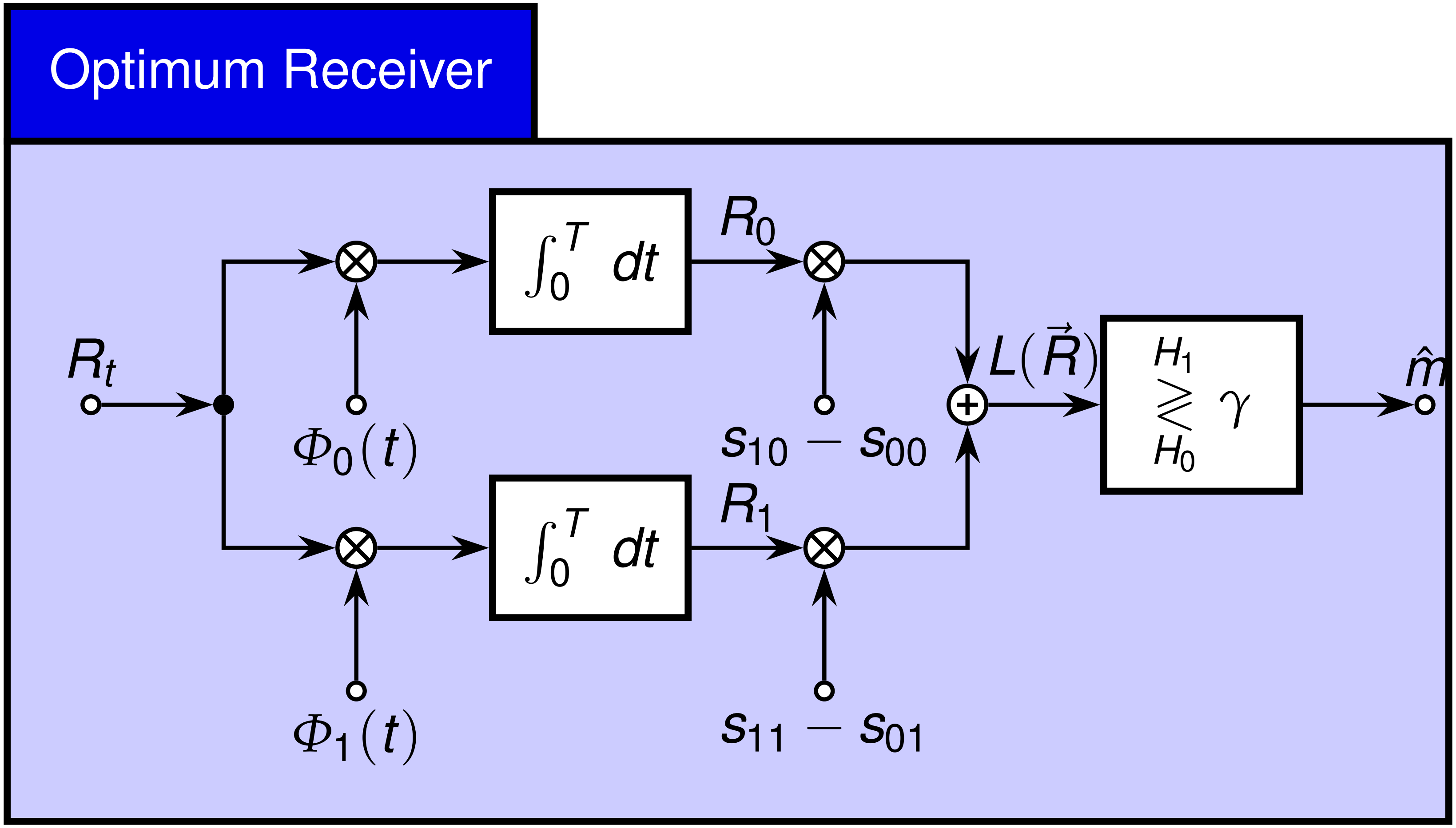
$$H_0\colon \vec{R} = \begin{pmatrix} s_{00} \\ s_{01} \end{pmatrix} + \begin{pmatrix} N_0 \\ N_1 \end{pmatrix} = \vec{s}_0 + \vec{N} \sim \mathrm{N}(\vec{s}_0, \frac{N_0}{2} I)$$

$$H_1\colon \vec{R} = \begin{pmatrix} s_{10} \\ s_{11} \end{pmatrix} + \begin{pmatrix} N_0 \\ N_1 \end{pmatrix} = \vec{s}_1 + \vec{N} \sim \mathrm{N}(\vec{s}_1, \frac{N_0}{2} I)$$

Optimum Receiver - Version 1

- MPE decision rule:
  1. Compute
  $$L(\vec{R}) = \langle \vec{R}, \vec{s}_1 - \vec{s}_0 \rangle.$$
  2. Compare to threshold:
  $$\gamma = \frac{N_0}{2} \ln(\pi_0/\pi_1) + \frac{\|\vec{s}_1\|^2 - \|\vec{s}_0\|^2}{2}$$
  3. Decision
  $$\text{If } L(\vec{R}) > \gamma \text{ decide } s_1(t) \text{ was sent.}$$
  $$\text{If } L(\vec{R}) < \gamma \text{ decide } s_0(t) \text{ was sent.}$$

Optimum Receiver - Version 1



Probability of Error

- The probability of error for this receiver is

$$\Pr\{e\} = \pi_0 Q\left(\frac{\|\vec{s}_0 - \vec{s}_1\|}{2\sqrt{\frac{N_0}{2}}} + \sqrt{\frac{N_0}{2}}\frac{\ln(\pi_0/\pi_1)}{\|\vec{s}_0 - \vec{s}_1\|}\right)$$
$$+ \pi_1 Q\left(\frac{\|\vec{s}_0 - \vec{s}_1\|}{2\sqrt{\frac{N_0}{2}}} - \sqrt{\frac{N_0}{2}}\frac{\ln(\pi_0/\pi_1)}{\|\vec{s}_0 - \vec{s}_1\|}\right)$$

- For the important special case of equally likely signals:

$$\Pr\{e\} = Q\left(\frac{\|\vec{s}_0 - \vec{s}_1\|}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\frac{\|\vec{s}_0 - \vec{s}_1\|}{\sqrt{2N_0}}.\right)$$

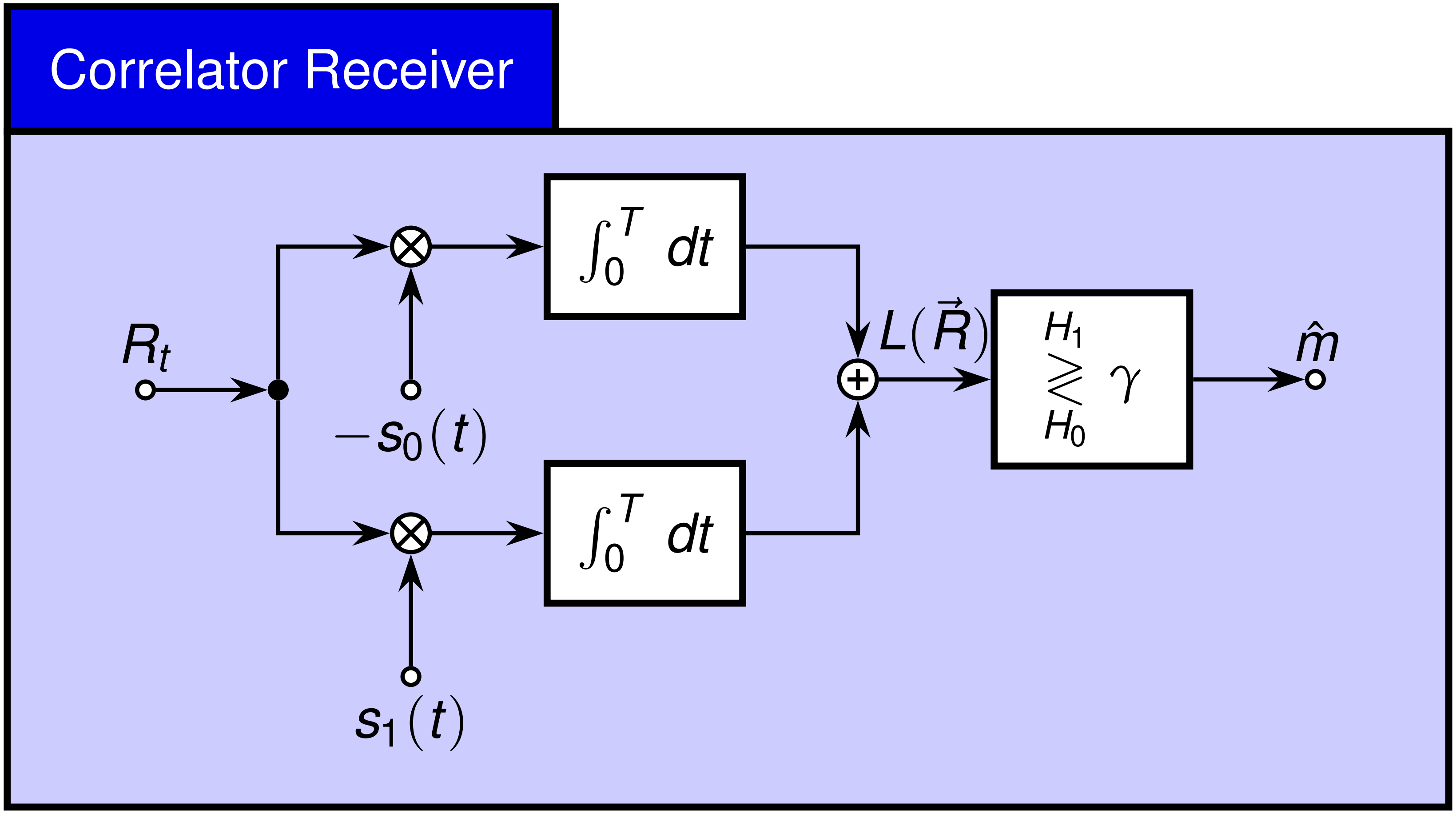
- This is the minimum probability of error achievable by *any* receiver.

Optimum Receiver - Version 2

- The optimum receiver derived above, computes the inner product
$$\langle \vec{R}, \vec{s}_1 - \vec{s}_0 \rangle.$$
- By Parseval's relationship, the inner product of the representation equals the inner product of the signals

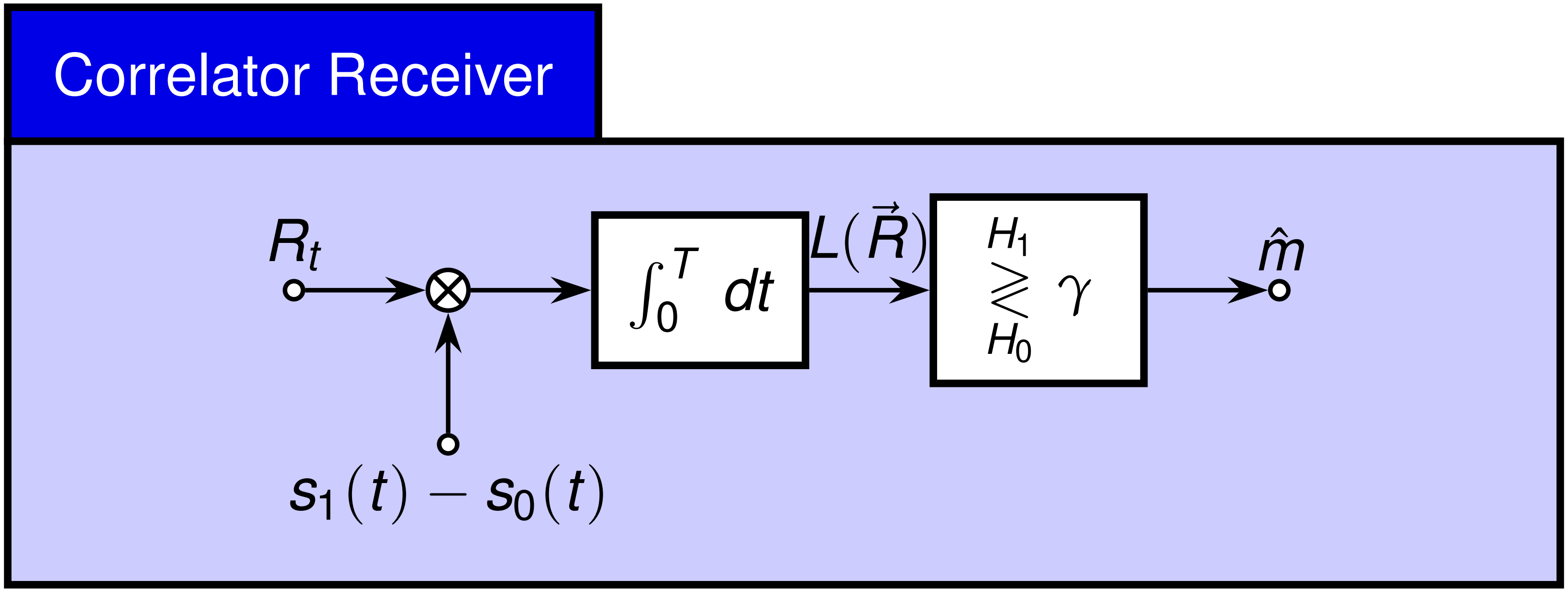
$$\begin{aligned}
\langle \vec{R}, \vec{s}_1 - \vec{s}_0 \rangle &= \langle R_t, s_1(t) - s_0(t) \rangle \\
&= \int_0^T R_t (s_1(t) - s_0(t))\, dt \\
&= \int_0^T R_t s_1(t)\, dt - \int_0^T R_t s_0(t))\, dt.
\end{aligned}$$

Optimum Receiver - Version 2



- Correlator receiver.

Optimum Receiver - Version 2a

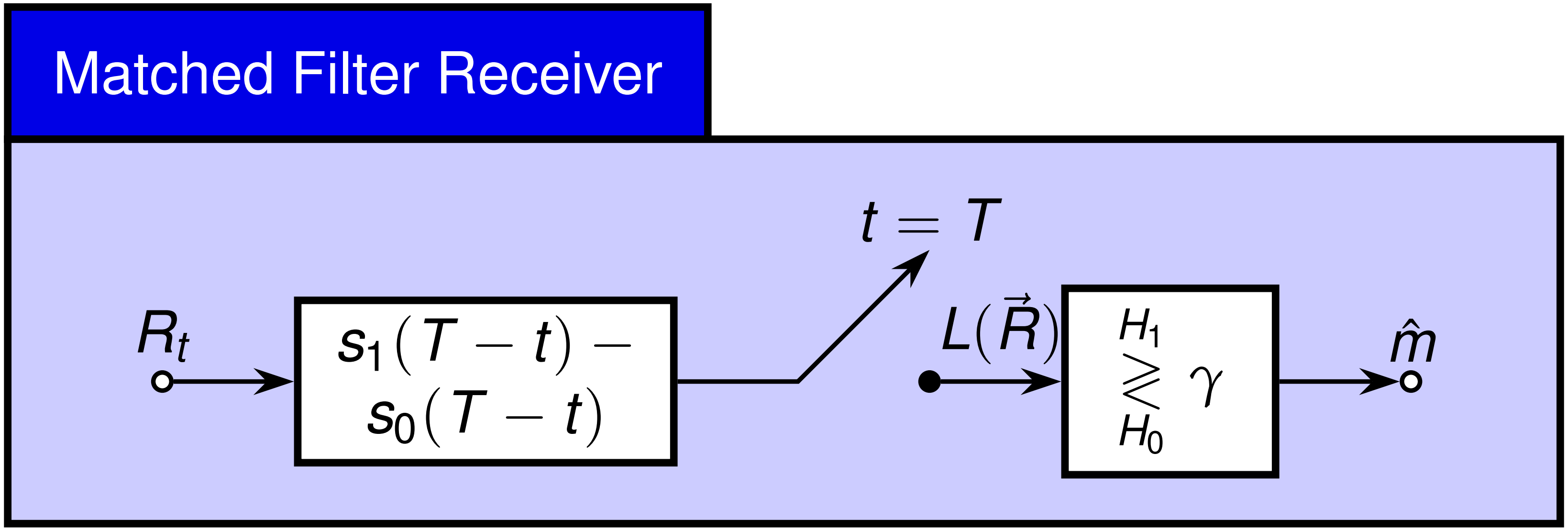


- The two correlators can be combined into a single correlator for an even simpler frontend.

Optimum Receiver - Version 3

- Yet another, important structure for the optimum receiver frontend results from the equivalence between *correlation* and *convolution followed by sampling*.
  - Convolution:
$$y(t) = x(t) * h(t) = \int_0^T x(\tau) h(t-\tau)\, d\tau$$
  - Sample at $t = T$:
$$y(T) = x(t) * h(t)|_{t=T} = \int_0^T x(\tau) h(T-\tau)\, d\tau$$
  - Let $g(t) = h(T-t)$ (and, thus, $h(t) = g(T-t)$):
$$\int_0^T x(t) g(t)\, dt = \int_0^T x(\tau) h(T-\tau)\, d\tau = x(t) * h(t)|_{t=T}.$$
- Correlating with $g(t)$ is equivalent to convolving with $h(t) = g(T-t)$, followed by symbol-rate sampling.

Optimum Receiver - Version 3



- The filter with impulse response $h(t) = s_1(T-t) - s_0(T-t)$ is called the matched filter for $s_1(t) - s_0(t)$.

Exercises: Optimum Receiver

- For each of the following signal sets:
  1. draw a block diagram of the MPE receiver,
  2. compute the value of the threshold in the MPE receiver,
  3. compute the probability of error for this receiver for $\pi_0 = \pi_1$,
  4. find basis functions for the signal set,
  5. illustrate the location of the signals in the signal space spanned by the basis functions,
  6. draw the decision boundary formed by the optimum receiver.

On-Off Keying

- Signal set:

$$\left.\begin{aligned} s_0(t) &= 0 \\ s_1(t) &= \sqrt{\frac{E}{T}} \end{aligned}\right\} \quad \text{for } 0 \leq t \leq T$$

  - This signal set is referred to as *On-Off Keying (OOK)* or *Amplitude Shift Keying (ASK)*.

Orthogonal Signalling

- Signal set:

$$s_0(t) = \begin{cases} \sqrt{\frac{E}{T}} & \text{for } 0 \leq t \leq \frac{T}{2} \\ -\sqrt{\frac{E}{T}} & \text{for } \frac{T}{2} \leq t \leq T \end{cases}$$

$$s_1(t) = \sqrt{\frac{E}{T}} \quad \text{for } 0 \leq t \leq T$$

- Alternatively:

$$\left.\begin{aligned} s_0(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_0 t) \\ s_1(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_1 t) \end{aligned}\right\} \quad \text{for } 0 \leq t \leq T$$

with $f_0 T$ and $f_1 T$ distinct integers.

  - This signal set is called *Frequency Shift Keying (FSK)*.

Antipodal Signalling

- Signal set:

$$\left.\begin{aligned} s_0(t) &= -\sqrt{\frac{E}{T}} \\ s_1(t) &= \sqrt{\frac{E}{T}} \end{aligned}\right\} \quad \text{for } 0 \leq t \leq T$$

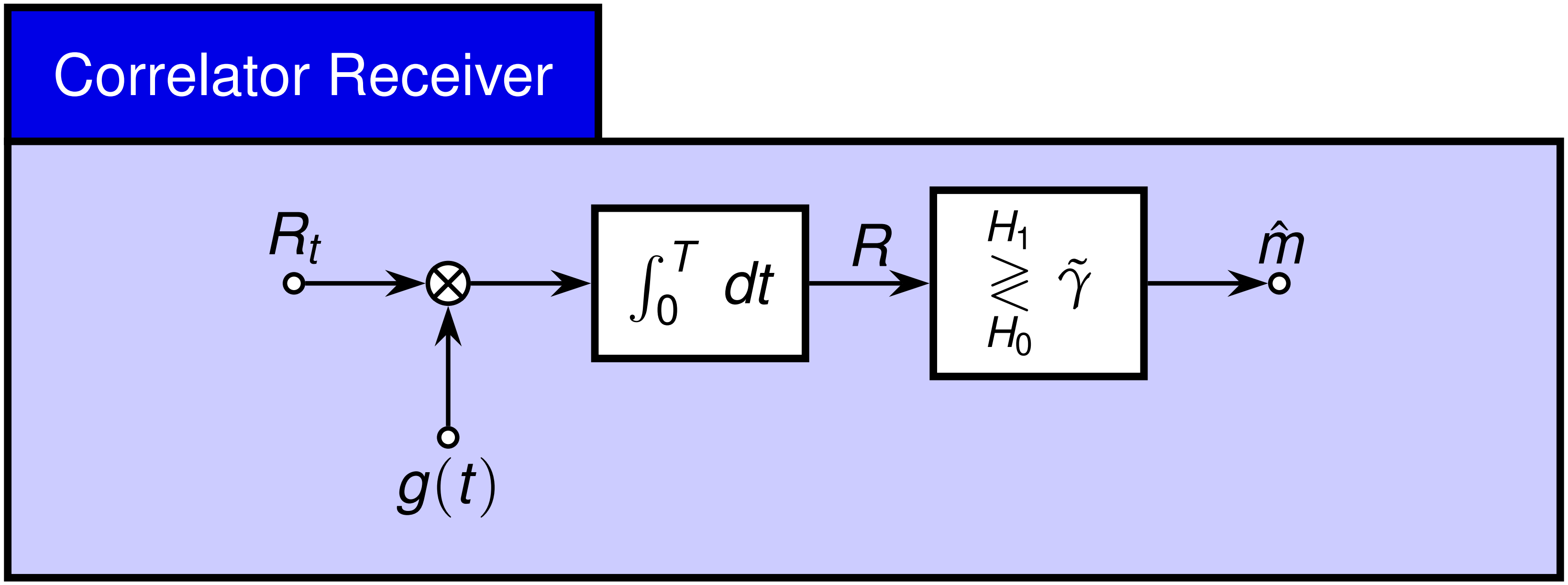
  - This signal set is referred to as *Antipodal Signalling*.
- Alternatively:

$$\left.\begin{aligned} s_0(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_0 t) \\ s_1(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_0 t + \pi) \end{aligned}\right\} \quad \text{for } 0 \leq t \leq T$$

  - This signal set is called *Binary Phase Shift Keying (BPSK)*.

Linear Receiver

- Consider a receiver with a "generic" linear frontend.



- We refer to these receivers as *linear receivers* because their frontend performs a linear transformation of the received signal.
  - Specifically, frontend computes $R = \langle R_t, g(t) \rangle$.

Linear Receiver

- **Objectives:**
  - derive general expressions for the conditional pdfs at the output R of the frontend,
  - derive general expressions for the error probability,
  - confirm that the optimum linear receiver correlates with $g(t) = s_1(t) - s_0(t)$,
    - i.e., the MPE receiver is also the best linear receiver.
- These results are useful for the analysis of arbitrary linear receivers.

Conditional Distributions

- Hypotheses:

$$H_0\colon R_t = s_0(t) + N_t$$
$$H_1\colon R_t = s_1(t) + N_t$$

  signals are observed for $0 \leq t \leq T$.
  - Priors are π_0 and π_1.
- Conditional distributions of $R = \langle R_t, g(t) \rangle$ are Gaussian:

$$H_0\colon R \sim \mathrm{N}(\underbrace{\langle s_0(t), g(t) \rangle}_{\mu_0}, \underbrace{\frac{N_0}{2} \|g(t)\|^2}_{\sigma^2})$$

$$H_1\colon R \sim \mathrm{N}(\underbrace{\langle s_1(t), g(t) \rangle}_{\mu_1}, \underbrace{\frac{N_0}{2} \|g(t)\|^2}_{\sigma^2})$$

MPE Decision Rule

- For the decision problem

$$H_0\colon R \sim \mathrm{N}(\underbrace{\langle s_0(t), g(t)\rangle}_{\mu_0}, \underbrace{\frac{N_0}{2}\|g(t)\|^2}_{\sigma^2})$$

$$H_1\colon R \sim \mathrm{N}(\underbrace{\langle s_1(t), g(t)\rangle}_{\mu_1}, \underbrace{\frac{N_0}{2}\|g(t)\|^2}_{\sigma^2})$$

the MPE decision rule is

$$R \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{\gamma}$$

with

$$\tilde{\gamma} = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma^2}{\mu_1 - \mu_0} \ln\left(\frac{\pi_0}{\pi_1}\right).$$

Probability of Error

- The probability of error, assuming $\pi_0 = \pi_1$, for this decision rule is

$$\Pr\{e\} = \mathrm{Q}\left(\frac{\mu_1 - \mu_0}{2\sigma}\right) = \mathrm{Q}\left(\frac{\langle s_1(t) - s_0(t), g(t)\rangle}{2\sqrt{\frac{N_0}{2}}\|g(t)\|}\right)$$

- **Question:** Which choice of $g(t)$ minimizes the probability of error?

Best Linear Receiver

- The probability of error is minimized when
$$\frac{\langle s_1(t) - s_0(t), g(t)\rangle}{2\sqrt{\frac{N_0}{2}}\|g(t)\|}$$
is maximized with respect to $g(t)$.
- We know from the Schwartz inequality that
$$\langle s_1(t) - s_0(t), g(t)\rangle \leq \|s_1(t) - s_0(t)\| \cdot \|g(t)\|$$
with equality if and only if $g(t) = c \cdot (s_1(t) - s_0(t))$, $c > 0$.
- Hence, to minimize probability of error, choose $g(t) = s_1(t) - s_0(t)$. Then,
$$\Pr\{e\} = \mathrm{Q}\left(\frac{\|s_1(t) - s_0(t)\|}{2\sqrt{\frac{N_0}{2}}}\right) = \mathrm{Q}\left(\frac{\|s_1(t) - s_0(t)\|}{\sqrt{2N_0}}\right)$$