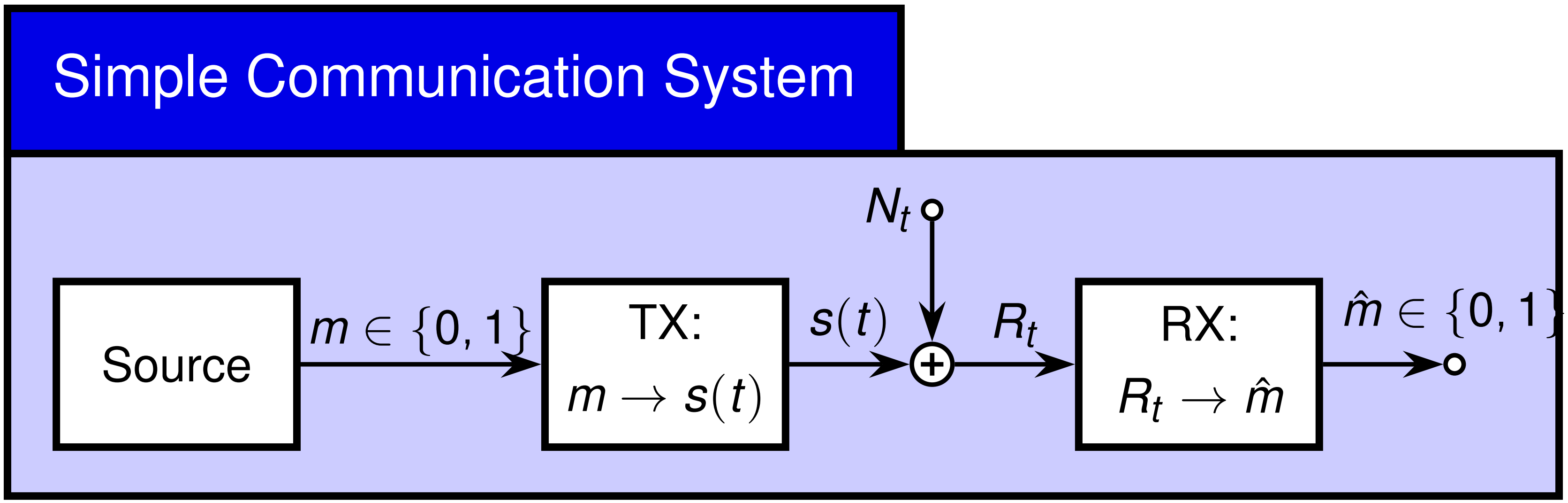
A Simple Communication System



- **Objectives:** For the above system
  - describe the optimum receiver and
  - find the probability of error for that receiver.

Assumptions

Noise: N_t is a white Gaussian noise process with spectral height $\frac{N_0}{2}$:

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau).$$

- Additive White Gaussian Noise (AWGN).

Source: characterized by the *a priori* probabilities

$$\pi_0 = \Pr\{m = 0\} \quad \pi_1 = \Pr\{m = 1\}.$$

- For this example, will assume $\pi_0 = \pi_1 = \frac{1}{2}$.

Assumptions (cont'd)

Transmitter: maps information bits m to signals:

$$m \to s(t) : \begin{cases} s_0(t) = \sqrt{\frac{E_b}{T}} & \text{if } m = 0 \\ s_1(t) = -\sqrt{\frac{E_b}{T}} & \text{if } m = 1 \end{cases}$$

for $0 \leq t \leq T$.

- Note that we are considering the transmission of a single bit.
- In AWGN channels, each bit can be considered in isolation.

Objective

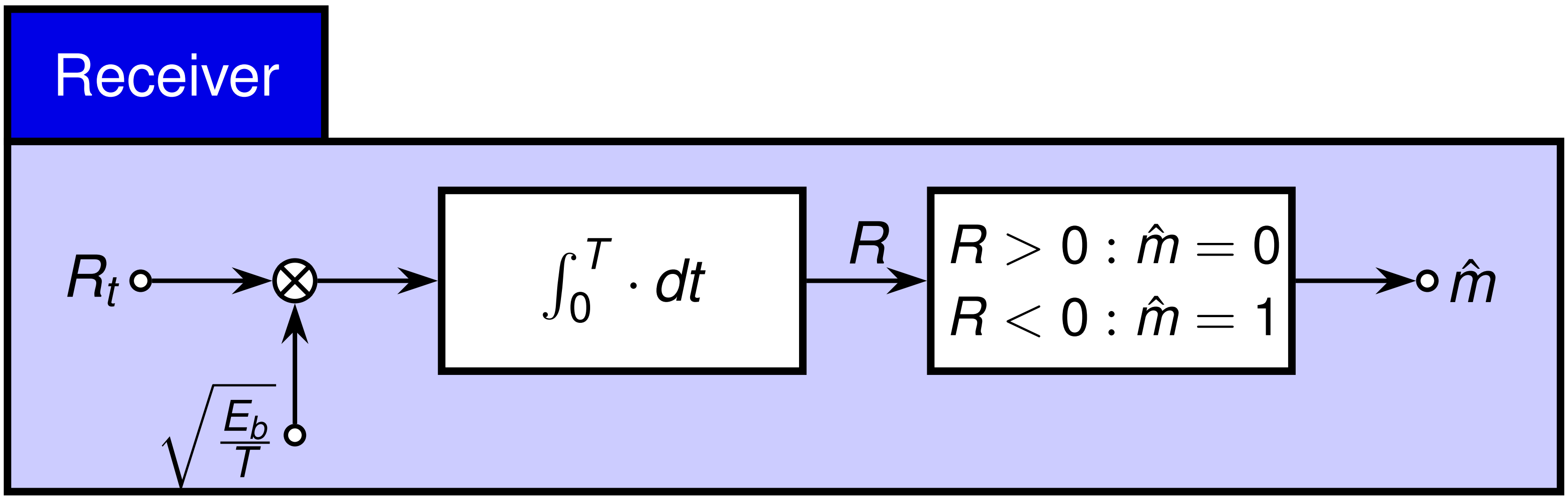
- In general, the objective is to find the receiver that minimizes the probability of error:

$$\begin{aligned}\Pr\{e\} &= \Pr\{\hat{m} \neq m\} \\ &= \pi_0 \Pr\{\hat{m} = 1 | m = 0\} + \pi_1 \Pr\{\hat{m} = 0 | m = 1\}.\end{aligned}$$

  - For this example, optimal receiver will be given (next slide).
- Also, compute the probability of error for the communication system.
  - That is the focus of this example.

Receiver

- We will see that the following receiver minimizes the probability of error for *this* communication system.



- **RX Frontend** computes $R = \int_0^T R_t \sqrt{\frac{E_b}{T}}\, dt = \langle R_t, s_0(t) \rangle$.
- **RX Backend** compares R to a threshold to arrive at decision $\hat{m}$.

Plan for Finding $\Pr\{e\}$

- Analysis of the receiver proceeds in the following steps:
  1. Find the *conditional* distribution of the output R from the receiver frontend.
     - Conditioning with respect to each of the possibly transmitted signals.
     - This boils down to finding conditional mean and variance of R.
  2. Find the conditional error probabilities $\Pr\{\hat{m} = 0 | m = 1\}$ and $\Pr\{\hat{m} = 1 | m = 0\}$.
     - Involves finding the probability that R exceeds a threshold.
  3. Total probability of error:

$$\Pr\{e\} = \pi_0 \Pr\{\hat{m} = 0 | m = 1\} + \pi_1 \Pr\{\hat{m} = 0 | m = 1\}.$$

Conditional Distribution of R

- There are two random effects that affect the received signal:
  - the additive white Gaussian noise N_t and
  - the random information bit m.
- By conditioning on m — thus, on $s(t)$ — randomness is caused by the noise only.
- Conditional on m, the output R of the receiver frontend is a Gaussian random variable:
  - N_t is a Gaussian random process; for given $s(t)$, $R_t = s(t) + N_t$ is a Gaussian random process.
  - The frontend performs a linear transformation of R_t: $R = \langle R_t, s_0(t) \rangle$.
- We need to find the conditional means and variances

Conditional Distribution of R

- The conditional means and variance of the frontend output R are

$$\mathbf{E}[R|m=0] = E_b \qquad \mathrm{Var}[R|m=0] = \frac{N_0}{2} E_b$$

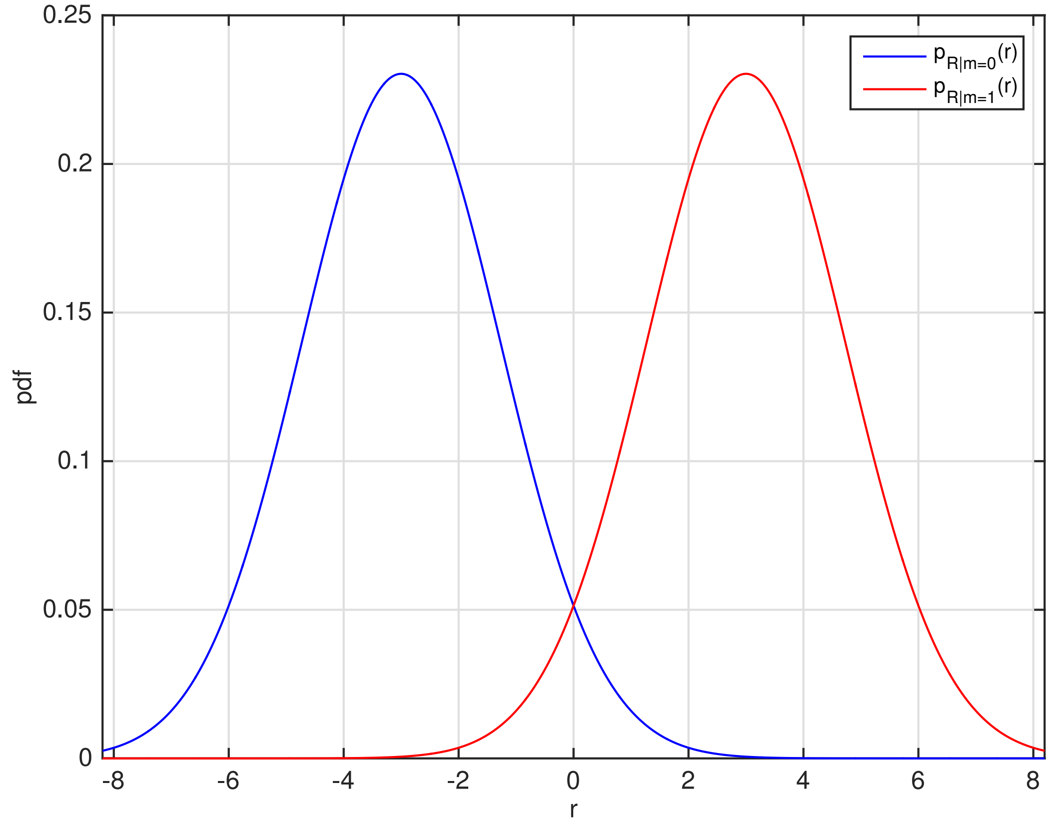
$$\mathbf{E}[R|m=1] = -E_b \qquad \mathrm{Var}[R|m=1] = \frac{N_0}{2} E_b$$

- Therefore, the conditional distributions of R are

$$p_{R|m=0}(r) \sim N(E_b, \frac{N_0}{2} E_b) \qquad p_{R|m=1}(r) \sim N(-E_b, \frac{N_0}{2} E_b)$$

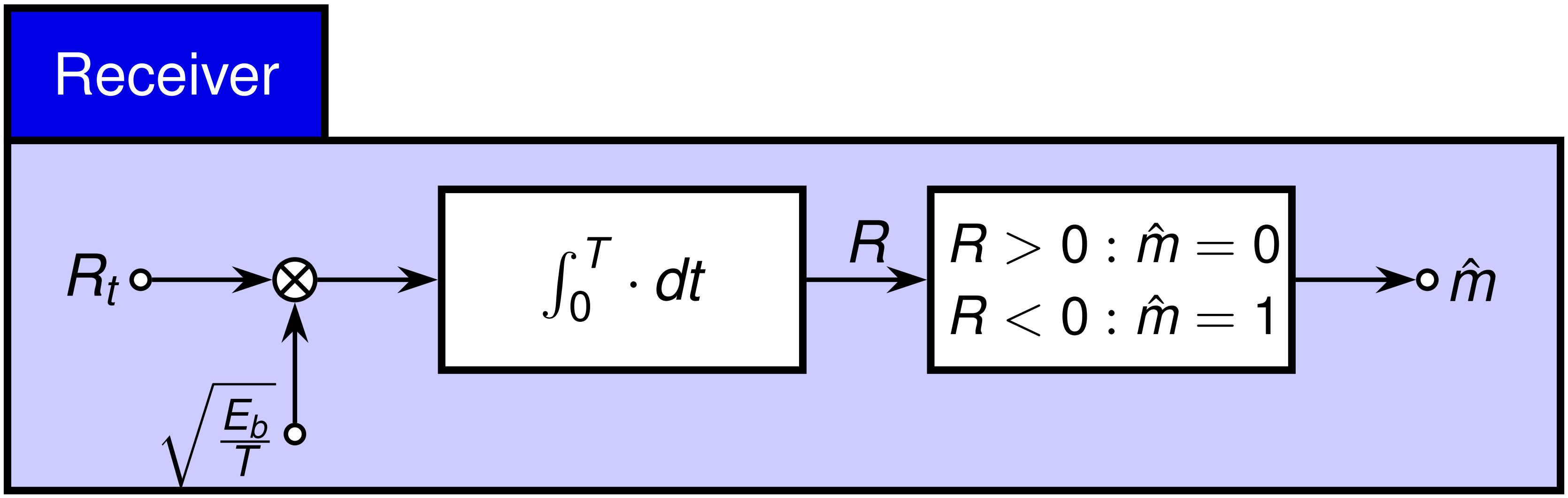
  - The two conditional distributions differ in the mean and have equal variances.

Conditional Distribution of R



- The two conditional pdfs are shown in the plot above, with
  - $E_b = 3$
  - $\frac{N_0}{2} = 1$

Conditional Probability of Error



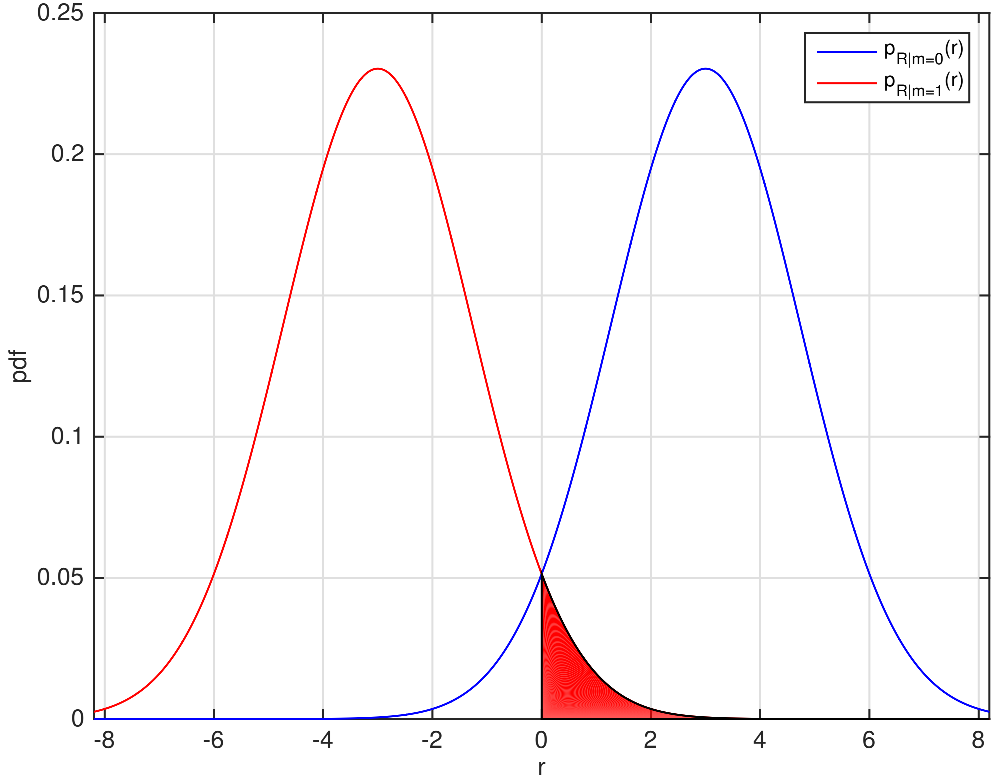
- The receiver backend decides:

$$\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0 \end{cases}$$

- Two conditional error probabilities:

$$\Pr\{\hat{m} = 0 | m = 1\} \quad \text{and} \quad \Pr\{\hat{m} = 1 | m = 0\}$$

Error Probability $\Pr\{\hat{m} = 0 | m = 1\}$

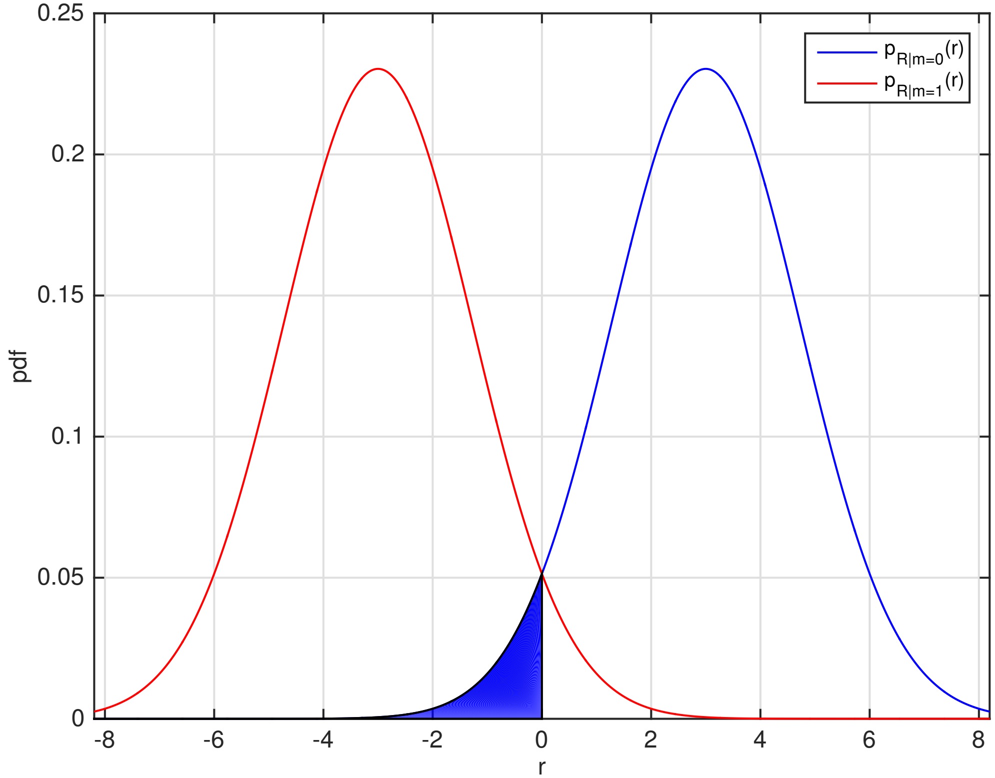


- Conditional error probability $\Pr\{\hat{m} = 0 | m = 1\}$ corresponds to shaded area.

$$\Pr\{\hat{m} = 0 | m = 1\} = \Pr\{R > 0 | m = 1\}$$

$$= \int_0^\infty p_{R|m=1}(r)\, dr = \mathrm{Q}\left(\sqrt{\frac{2E_b}{N0}}\right)$$

Error Probability $\Pr\{\hat{m} = 1 | m = 0\}$



- Conditional error probability $\Pr\{\hat{m} = 1 | m = 0\}$ corresponds to shaded area.

$$\Pr\{\hat{m} = 1 | m = 0\} = \Pr\{R < 0 | m = 0\}$$

$$= \int_{-\infty}^{0} p_{R|m=0}(r)\, dr = Q\left(\sqrt{\frac{2E_b}{N0}}\right).$$

Average Probability of Error

- The (average) probability of error is the average of the two conditional probabilities of error.
  - The average is weighted by the a priori probabilities π_0 and π_1.
- Thus,

$$\Pr\{e\} = \pi_0 \Pr\{\hat{m} = 1 | m = 0\} + \pi_1 \Pr\{\hat{m} = 0 | m = 1\}.$$

- With the above conditional error probabilities and equal priors $\pi_0 = \pi_1 = \frac{1}{2}$

$$\Pr\{e\} = \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N0}}\right) = Q\left(\sqrt{\frac{2E_b}{N0}}\right).$$

- Note that the error probability depends on the ratio $\frac{E_b}{N_0}$,
  - where E_b is the energy of signals $s_0(t)$ and $s_1(t)$.
  - This ratio is referred to as the signal-to-noise ratio.

Exercise - Compute Probability of Error

- Compute the probability of error for the example system if the only change in the system is that signals $s_0(t)$ and $s_1(t)$ are changed to triangular signals:

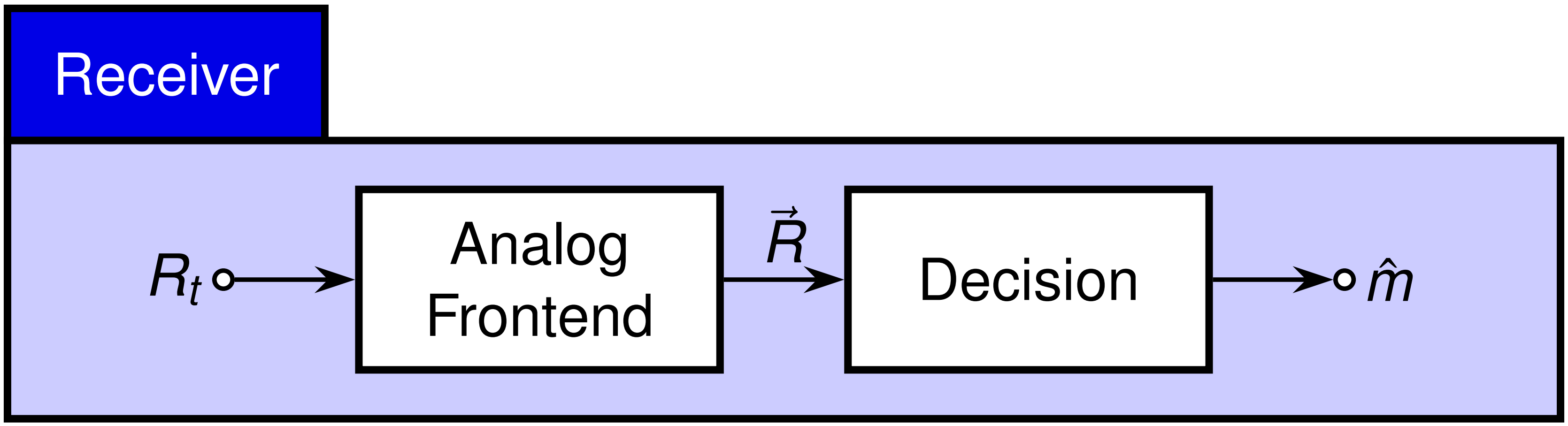
$$s_0(t) = \begin{cases} \frac{2A}{T} \cdot t & \text{for } 0 \leq t \leq \frac{T}{2} \\ 2A - \frac{2A}{T} \cdot t & \text{for } \frac{T}{2} \leq t \leq T \\ 0 & \text{else} \end{cases} \qquad s_1(t) = -s_0(t)$$

with $A = \sqrt{\frac{3E_b}{T}}$.

- **Answer:**

$$\Pr\{e\} = Q\left(\sqrt{\frac{3E_b}{2N_0}}\right)$$

Structure of a Generic Receiver



- Receivers consist of:
  - an *analog frontend*: maps observed signal R_t to decision statistic $\vec{R}$.
  - *decision device*: determines which symbol $\hat{m}$ was sent based on observation of $\vec{R}$.
- Optimum design of decision device will be considered first.

Problem Setup

- **Given:**
  - a random vector $\vec{R} \in \mathbb{R}^n$ of observations and
  - hypotheses, H_0 and H_1, providing statistical models for $\vec{R}$:

$$H_0\colon \vec{R} \sim p_{\vec{R}|H_0}(\vec{r}|H_0)$$

$$H_1\colon \vec{R} \sim p_{\vec{R}|H_1}(\vec{r}|H_1)$$

  with known *a priori* probabilities $\pi_0 = \Pr\{H_0\}$ and $\pi_1 = \Pr\{H_1\}$ ($\pi_0 + \pi_1 = 1$).

- **Problem:** Decide which of the two hypotheses is best supported by the observation $\vec{R}$.
  - Specific objective: minimize the probability of error

$$\begin{aligned}\Pr\{e\} &= \Pr\{\text{decide } H_0 \text{ when } H_1 \text{ is true}\} \\ &\quad + \Pr\{\text{decide } H_1 \text{ when } H_0 \text{ is true}\} \\ &= \Pr\{\text{decide } H_0|H_1\}\Pr\{H_1\} + \Pr\{\text{decide } H_1|H_0\}\Pr\{H_0\}\end{aligned}$$

Generic Decision Rule

- The decision device performs a mapping that assigns a decision, H_0 or H_1, to each possible observation $\vec{R} \in \mathbb{R}^n$.
- A generic way to realize such a mapping is:
  - partition the space of all possible observations, $\mathbb{R}^n$, into two disjoint, complementary decision regions Γ_0 and Γ_1:
  $$\Gamma_0 \cup \Gamma_1 = \mathbb{R}^n \text{ and } \Gamma_0 \cap \Gamma_1 = \emptyset.$$
  - **Decision Rule:**
  $$\text{If } \vec{R} \in \Gamma_0\text{: decide } H_0$$
  $$\text{If } \vec{R} \in \Gamma_1\text{: decide } H_1$$

Probability of Error

- The probability of error can now be expressed in terms of the decision regions Γ_0 and Γ_1:

$$\begin{aligned}\Pr\{e\} &= \Pr\{\text{decide } H_0|H_1\}\Pr\{H_1\} + \Pr\{\text{decide } H_1|H_0\}\Pr\{H_0\} \\ &= \pi_1 \int_{\Gamma_0} p_{\vec{R}|H_1}(\vec{r}|H_1)\, d\vec{r} + \pi_0 \int_{\Gamma_1} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r}\end{aligned}$$

- Our objective becomes to find the decision regions Γ_0 and Γ_1 that minimize the probability of error.

Probability of Error

- Since $\Gamma_0 \cup \Gamma_1 = \mathbb{R}^n$ it follows that $\Gamma_1 = \mathbb{R}^n \setminus \Gamma_0$

$$\begin{aligned}
\Pr\{e\} &= \pi_1 \int_{\Gamma_0} p_{\vec{R}|H_1}(\vec{r}|H_1)\, d\vec{r} + \pi_0 \int_{\mathbb{R}^n \setminus \Gamma_0} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r} \\
&= \pi_0 \int_{\mathbb{R}^n} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r} \\
&\quad + \int_{\Gamma_0} (\pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1) - \pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0))\, d\vec{r} \\
&= \pi_0 - \int_{\Gamma_0} (\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1))\, d\vec{r}.
\end{aligned}$$

- $\Pr\{e\}$ is minimized by chosing Γ_0 to contain all $\vec{r}$ for which the integrand $(\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)) < 0$.

Minimum $\Pr\{e\}$ (MPE) Decision Rule

- Thus, the decision region Γ_0 that minimizes the probability of error is given by:

$$\begin{aligned}\Gamma_0 &= \left\{\vec{r} : (\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)) > 0\right\} \\ &= \left\{\vec{r} : \pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) > \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1))\right\} \\ &= \left\{\vec{r} : \frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} < \frac{\pi_0}{\pi_1}\right\}\end{aligned}$$

- The decision region Γ_1 follows

$$\Gamma_1 = \Gamma_0^C = \left\{\vec{r} : \frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} > \frac{\pi_0}{\pi_1}\right\}$$

Likelihood Ratio

- The MPE decision rule can be written as

$$\text{If } \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \begin{cases} > \frac{\pi_0}{\pi_1} & \text{decide } H_1 \\ < \frac{\pi_0}{\pi_1} & \text{decide } H_0 \end{cases}$$

  - **Notation:**

$$\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

- The ratio of conditional density functions

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}$$

is called the likelihood ratio.

Log-Likelihood Ratio

- Many of the densities of interest are exponential functions (e.g., Gaussian).
- For these densities, it is advantageous to take the log of both sides of the decision rule.
  - **Important:** This does not change the decision rule because the logarithm is monotonically increasing!
- The MPE decision rule can be written as:

$$L(\vec{R}) = \ln\left(\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\pi_0}{\pi_1}\right)$$

- $L(\vec{R}) = \ln(\Lambda(\vec{R}))$ is called the log-likelihood ratio.