Parsevals Relationship

- **Parsevals Theorem:** If vectors x and y are represented with respect to an orthonormal basis $\{\Phi_n\}$ by $\{X_n\}$ and $\{Y_n\}$, respectively, then

$$\langle x, y \rangle = \sum_{n=1}^{\infty} X_n \cdot Y_n$$

Parsevals Relationship

- Parsevals theorem implies

$$\|x\|^2 = \sum_{n=1}^{\infty} X_n^2$$

and

$$\|x-y\|^2 = \sum_{n=1}^{\infty} |X_n - Y_n|^2$$

- Inner products, norms, and distances can be computed using vectors or their representations; the results are the same.

Back to the Projection Theorem

- We claimed earlier that the projection theorem is particularly useful when the subspace $\mathcal{L}$ is structured.
- Specifically, let $\mathcal{L}$ be a subspace of $\mathcal{S}$ spanned by a (usually finite) orthonormal basis $\{\Phi_n\}_{n=0}^{N-1}$.
  - Note that $\{\Phi_n\}_{n=0}^{N-1}$ is **not** a complete basis for $\mathcal{S}$.
  - There are $x \in \mathcal{S}$ that cannot be represented by this basis.
- Then, the projection $y \in \mathcal{L}$ of a vector $x \in \mathcal{S}$ is simply

$$y = \sum_{n=0}^{N-1} Y_n \Phi_n \text{ with } Y_n = \langle x, \Phi_n \rangle.$$

- **Examples:**
  - Band-limited Fourier series expansion
  - Polynomial regression with Legendre polynomials

Exercise: Orthonormal Basis

- Show that for orthonormal basis $\{\Phi_n\}$, the representation X_n of x is obtained by projection

$$\langle x, \Phi_n \rangle = X_n$$

  - **Hint:** You need to find

$$\hat{X}_n = \arg\min_{X_n} \|x - X_n\Phi_n - \sum_{m \neq n} X_m \Phi_m\|^2$$

- Show that Parsevals theorem is true.

The Gram-Schmidt Procedure

- An arbitrary basis $\{\Phi_n\}$ can be converted into an orthonormal basis $\{\Psi_n\}$ using an algorithm known as the Gram-Schmidt procedure:

Step 1: $\Psi_1 = \frac{\Phi_1}{\|\Phi_1\|}$ (normalize Φ_1)

Step 2 (a): $\tilde{\Psi}_2 = \Phi_2 - \langle \Phi_2, \Psi_1 \rangle \cdot \Psi_1$ (make $\tilde{\Psi}_2 \perp \Psi_1$)

Step 2 (b): $\Psi_2 = \frac{\tilde{\Psi}_2}{\|\tilde{\Psi}_2\|}$

$$\vdots$$

Step k (a): $\tilde{\Psi}_k = \Phi_k - \sum_{n=1}^{k-1} \langle \Phi_k, \Psi_n \rangle \cdot \Psi_n$

Step k (b): $\Psi_k = \frac{\tilde{\Psi}_k}{\|\tilde{\Psi}_k\|}$

  - Whenever $\tilde{\Psi}_n = 0$, the basis vector is omitted.

Gram-Schmidt Procedure

- Note:
  - By construction, $\{\Psi\}$ is an orthonormal set of vectors.
  - If the orginal basis $\{\Phi\}$ is complete, then $\{\Psi\}$ is also complete.
    - The Gram-Schmidt construction implies that every Φ_n can be represented in terms of Ψ_m, with $m = 1, \ldots, n$.
- Because
  - any basis can be normalized (using the Gram-Schmidt procedure) and
  - the benefits of orthonormal bases when computing the representation of a vector

  a basis is usually assumed to be orthonormal.

Exercise: Gram-Schmidt Procedure

- The following three basis functions are given

$$\Phi_1(t) = I_{[0,\frac{T}{2}]}(t) \quad \Phi_2(t) = I_{[0,T]}(t) \quad \Phi_3(t) = I_{[\frac{T}{2},T]}(t)$$

  where $I_{[a,b]}(t) = 1$ for $a \leq t \leq b$ and zero otherwise.

  1. Compute an *orthonormal* basis from the above basis functions.
  2. Compute the representation of $\Phi_n(t)$, $n = 1, 2, 3$ with respect to this orthonormal basis.
  3. Compute $\|\Phi_1(t)\|$ and $\|\Phi_2(t) - \Phi_3(t)\|$

Answers for Exercise

1. Orthonormal bases:

$$\Psi_1(t) = \sqrt{\frac{2}{T}} I_{[0,\frac{T}{2}]}(t) \quad \Psi_2(t) = \sqrt{\frac{2}{T}} I_{[\frac{T}{2},T]}(t)$$

2. Representations:

$$\phi_1 = \begin{pmatrix} \sqrt{\frac{T}{2}} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{\frac{T}{2}} \\ \sqrt{\frac{T}{2}} \end{pmatrix} \quad \begin{pmatrix} 0 \\ \sqrt{\frac{T}{2}} \end{pmatrix}$$

3. Distances: $\|\Phi_1(t)\| = \sqrt{\frac{T}{2}}$ and $\|\Phi_2(t) - \Phi_3(t)\| = \sqrt{\frac{T}{2}}$.

A Hilbert Space for Random Processes

- A vector space for random processes X_t that is analogous to $L^2(a,b)$ is of greatest interest to us.
  - This vector space contains random processes that satisfy, i.e.,
  $$\int_a^b \mathbf{E}[X_t^2]\, dt < \infty.$$
  - **Inner Product:** cross-correlation
  $$\mathbf{E}[\langle X_t, Y_t \rangle] = \mathbf{E}[\int_a^b X_t Y_t\, dt].$$
  - Fact: This vector space is separable; therefore, an orthonormal basis $\{\Phi\}$ exists.

A Hilbert Space for Random Processes

- (con't)
  - **Representation:**

$$X_t = \sum_{n=1}^{\infty} X_n \Phi_n(t) \quad \text{for } a \leq t \leq b$$

with

$$X_n = \langle X_t, \Phi_n(t) \rangle = \int_a^b X_t \Phi_n(t)\, dt.$$

    - Note that X_n is a *random variable*.
  - For this to be a valid Hilbert space, we must interprete equality of processes X_t and Y_t in the mean squared sense, i.e.,

$$X_t = Y_t \text{ means } \mathbf{E}[|X_t - Y_t|^2] = 0.$$

Karhunen-Loeve Expansion

- **Goal:** Choose an orthonormal basis $\{\Phi\}$ such that the representation $\{X_n\}$ of the random process X_t consists of *uncorrelated random variables*.
  - The resulting representation is called Karhunen-Loeve expansion.
- Thus, we want

$$\mathbf{E}[X_n X_m] = \mathbf{E}[X_n]\mathbf{E}[X_m] \quad \text{for } n \neq m.$$

Karhunen-Loeve Expansion

- It can be shown, that for the representation $\{X_n\}$ to consist of uncorrelated random variables, the orthonormal basis vectors $\{\Phi\}$ must satisfy

$$\int_a^b K_X(t,u) \cdot \Phi_n(u)\,du = \lambda_n \Phi_n(t)$$

  - where $\lambda_n = \text{Var}[X_n]$.
  - $\{\Phi_n\}$ and $\{\lambda_n\}$ are the eigenfunctions and eigenvalues of the autocovariance function $K_X(t,u)$, respectively.

Example: Wiener Process

- For the Wiener Process, the autocovariance function is

$$K_X(t,u) = R_X(t,u) = \sigma^2 \min(t,u).$$

- It can be shown that

$$\begin{aligned}\Phi_n(t) &= \sqrt{\frac{2}{T}} \sin((n-\frac{1}{2})\pi\frac{t}{T}) \\ \lambda_n &= \left(\frac{\sigma T}{(n-\frac{1}{2})\pi}\right)^2\end{aligned} \quad \text{for } n = 1, 2, \ldots.$$

Properties of the K-L Expansion

- The eigenfunctions of the autocovariance function form a complete basis.
- If X_t is Gaussian, then the representation $\{X_n\}$ is a vector of independent, Gaussian random variables.
- For white noise, $K_X(t,u) = \frac{N_0}{2}\delta(t-u)$. Then, the eigenfunctions must satisfy

$$\int \frac{N_0}{2}\delta(t-u)\Phi(u)\,du = \lambda\Phi(t).$$

  - Any orthonormal set of bases $\{\Phi\}$ satisfies this condition!
  - Eigenvalues λ are all equal to $\frac{N_0}{2}$.
  - If X_t is white, Gaussian noise then the representation $\{X_n\}$ are independent, identically distributed random variables.
    - zero mean
    - variance $\frac{N_0}{2}$