Separable Vector Spaces

- **Definition:** A Hilbert space $\mathcal{H}$ is said to be separable if there exists a set of vectors $\{\Phi_n\}$, $n = 1, 2, \ldots$ that are elements of $\mathcal{H}$ and such that every element $x \in \mathcal{H}$ can be expressed as

$$x = \sum_{n=1}^{\infty} X_n \Phi_n.$$

  - The coefficients X_n are scalars associated with vectors Φ_n.
  - *Equality* is taken to mean

$$\lim_{n \to \infty} \left\| x - \sum_{n=1}^{\infty} X_n \Phi_n \right\|^2 = 0.$$

Representation of a Vector

- The set of vectors $\{\Phi_n\}$ is said to be complete if the above is valid for every $x \in \mathcal{H}$.
- A complete set of vectors $\{\Phi_n\}$ is said to form a basis for $\mathcal{H}$.
- **Definition:** The representation of the vector x (with respect to the basis $\{\Phi_n\}$) is the sequence of coefficients $\{X_n\}$.
- **Definition:** The number of vectors Φ_n that is required to express every element x of a separable vector space is called the dimension of the space.

Example: Length-N column Vectors

- The space $\mathbb{R}^N$ is separable and has dimension N.
  - Basis vectors ($m = 1, \ldots, N$):

$$\Phi_m = e_m = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ the 1 occurs on the } m\text{-th row}$$

  - There are N basis vectors; dimension is N.

Example: Length-N column Vectors — continued

- (con't)
  - For any vector $x \in \mathbb{R}^N$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \sum_{m=1}^{N} x_m e_m$$

Examples: L^2

- **Fourier Bases:** The following is a complete basis for $L^2(0, T)$

$$\left.\begin{aligned} \Phi_{2n}(t) &= \sqrt{\frac{2}{T}} \cos(2\pi nt/T) \\ \Phi_{2n+1}(t) &= \sqrt{\frac{2}{T}} \sin(2\pi nt/T) \end{aligned}\right. \quad n = 0, 1, 2, \ldots$$

  - This implies that $L^2(0, T)$ is a separable vector space.
  - $L^2(0, T)$ is infinite-dimensional.

Examples: L^2

- **Piecewise Linear Signals:** The set of vectors (signals)

$$\Phi_n(t) = \begin{cases} \frac{1}{\sqrt{T}} & (n-1)T \leq t < nT \\ 0 & \text{else} \end{cases}$$

is **not** a basis for $L^2(0, \infty)$.
  - Only piecewise constant signals can be represented.

Orthonormal Bases

- **Definition:** A basis for a separable vector space is an orthonormal basis if the elements of the vectors that constitute the basis satisfy
  1. $\langle \Phi_n, \Phi_m \rangle = 0$ for all $n \neq m$. (*ortho*gonal)
  2. $\|\Phi_n\| = 1$, for all $n = 1, 2, \ldots$ (*normal*ized)
- Note:
  - Not every basis is orthonormal.
    - We will see shortly, every basis can be turned into an orthonormal basis.
  - Not every set of orthonornal vectors constitutes a basis.
    - Example: Piecewise Linear Signals.

Representation with Orthonormal Basis

- An orthonormal basis is much prefered over an arbitrary basis because the representation of vector x is very easy to compute.
- The representation $\{X_n\}$ of a vector x

$$x = \sum_{n=1}^{\infty} X_n \Phi_n$$

with respect to an orthonormal basis $\{\Phi_n\}$ is computed using

$$X_n = \langle x, \Phi_n \rangle.$$

The representation X_n is obtained by projecting x onto the basis vector Φ_n!

  - In contrast, when bases are not orthonormal, finding the representation of x requires solving a system of linear equations.

Parsevals Relationship

- **Parsevals Theorem:** If vectors x and y are represented with respect to an orthonormal basis $\{\Phi_n\}$ by $\{X_n\}$ and $\{Y_n\}$, respectively, then

$$\langle x, y \rangle = \sum_{n=1}^{\infty} X_n \cdot Y_n$$

- With $x = y$, Parsevals theorem implies

$$\|x\|^2 = \sum_{n=1}^{\infty} X_n^2$$

and

$$\|x - y\|^2 = \sum_{n=1}^{\infty} |X_n - Y_n|^2$$

Exercise: Orthonormal Basis

- Show that for orthonormal basis $\{\Phi_n\}$, the representation X_n of x is obtained by projection

$$\langle x, \Phi_n \rangle = X_n$$

- Show that Parsevals theorem is true.

The Gram-Schmidt Procedure

- An arbitrary basis $\{\Phi_n\}$ can be converted into an orthonormal basis $\{\Psi_n\}$ using an algorithm known as the Gram-Schmidt procedure:

  Step 1: $\Psi_1 = \frac{\Phi_1}{\|\Phi_1\|}$ (normalize Φ_1)

  Step 2 (a): $\tilde{\Psi}_2 = \Phi_2 - \langle \Phi_2, \Psi_1 \rangle \cdot \Psi_1$ (make $\tilde{\Psi}_2 \perp \Psi_1$)

  Step 2 (b): $\Psi_2 = \frac{\tilde{\Psi}_2}{\|\tilde{\Psi}_2\|}$

  $\vdots$

  Step k (a): $\tilde{\Psi}_k = \Phi_k - \sum_{n=1}^{k-1} \langle \Phi_k, \Psi_n \rangle \cdot \Psi_n$

  Step k (b): $\Psi_k = \frac{\tilde{\Psi}_k}{\|\tilde{\Psi}_k\|}$

  - Whenever $\tilde{\Psi}_n = 0$, the basis vector is omitted.

Gram-Schmidt Procedure

- Note:
  - By construction, $\{\Psi\}$ is an orthonormal set of vectors.
  - If the orginal basis $\{\Phi\}$ is complete, then $\{\Psi\}$ is also complete.
    - The Gram-Schmidt construction implies that every Φ_n can be represented in terms of Ψ_m, with $m = 1, \ldots, n$.
- Because
  - any basis can be normalized (using the Gram-Schmidt procedure) and
  - the benefits of orthonormal bases when computing the representation of a vector

  a basis is usually assumed to be orthonormal.

Exercise: Gram-Schmidt Procedure

- The following three basis functions are given

$$\Phi_1(t) = I_{[0,\frac{T}{2}]}(t) \quad \Phi_2(t) = I_{[0,T]}(t) \quad \Phi_3(t) = I_{[\frac{T}{2},T]}(t)$$

where $I_{[a,b]}(t) = 1$ for $a \leq t \leq b$ and zero otherwise.

1. Compute an *orthonormal* basis from the above basis functions.
2. Compute the representation of $\Phi_n(t)$, $n = 1, 2, 3$ with respect to this orthonormal basis.
3. Compute $\|\Phi_1(t)\|$ and $\|\Phi_2(t) - \Phi_3(t)\|$

Answers for Exercise

1. Orthonormal bases:

$$\Psi_1(t) = \sqrt{\frac{2}{T}} I_{[0,\frac{T}{2}]}(t) \quad \Psi_2(t) = \sqrt{\frac{2}{T}} I_{[\frac{T}{2},T]}(t)$$

2. Representations:

$$\phi_1 = \begin{pmatrix} \sqrt{\frac{T}{2}} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{\frac{T}{2}} \\ \sqrt{\frac{T}{2}} \end{pmatrix} \quad \begin{pmatrix} 0 \\ \sqrt{\frac{T}{2}} \end{pmatrix}$$

3. Distances: $\|\Phi_1(t)\| = \sqrt{\frac{T}{2}}$ and $\|\Phi_2(t) - \Phi_3(t)\| = \sqrt{\frac{T}{2}}$.

A Hilbert Space for Random Processes

- A vector space for random processes X_t that is analogous to $L^2(a, b)$ is of greatest interest to us.
  - This vector space contains random processes that satisfy, i.e.,
$$\int_a^b \mathbf{E}[X_t^2]\, dt < \infty.$$
  - **Inner Product:** cross-correlation
$$\mathbf{E}[\langle X_t, Y_t \rangle] = \mathbf{E}[\int_a^b X_t Y_t \, dt].$$
  - This vector space is separable; therefore, an orthonormal basis $\{\Phi\}$ exists.

A Hilbert Space for Random Processes

- (con't)
  - **Representation:**

$$X_t = \sum_{n=1}^{\infty} X_n \Phi_n(t) \quad \text{for } a \leq t \leq b$$

with

$$X_n = \langle X_t, \Phi_n(t) \rangle = \int_a^b X_t \Phi_n(t)\, dt.$$

    - Note that X_n is a *random variable*.
  - For this to be a valid Hilbert space, we must interprete equality of processes X_t and Y_t in the mean squared sense, i.e.,

$$X_t = Y_t \text{ means } \mathbf{E}[|X_t - Y_t|^2] = 0.$$

Karhunen-Loeve Expansion

- **Goal:** Choose an orthonormal basis $\{\Phi\}$ such that the representation $\{X_n\}$ of the random process X_t consists of *uncorrelated random variables*.
  - The resulting representation is called Karhunen-Loeve expansion.
- Thus, we want

$$\mathbf{E}[X_n X_m] = \mathbf{E}[X_n]\mathbf{E}[X_m] \quad \text{for } n \neq m.$$

Karhunen-Loeve Expansion

- It can be shown, that for the representation $\{X_n\}$ to consist of uncorrelated random variables, the orthonormal basis vectors $\{\Phi\}$ must satisfy

$$\int_a^b K_X(t,u) \cdot \Phi_n(u)\, du = \lambda_n \Phi_n(t)$$

  - where $\lambda_n = \text{Var}[X_n]$.
  - $\{\Phi_n\}$ and $\{\lambda_n\}$ are the eigenfunctions and eigenvalues of the autocovariance function $K_X(t,u)$, respectively.

Example: Wiener Process

- For the Wiener Process, the autocovariance function is

$$K_X(t,u) = R_X(t,u) = \sigma^2 \min(t,u).$$

- It can be shown that

$$\begin{aligned} \Phi_n(t) &= \sqrt{\frac{2}{T}} \sin((n-\frac{1}{2})\pi\frac{t}{T}) \\ \lambda_n &= \left(\frac{\sigma T}{(n-\frac{1}{2})\pi}\right)^2 \end{aligned} \quad \text{for } n = 1, 2, \ldots.$$

Properties of the K-L Expansion

- The eigenfunctions of the autocovariance function form a complete basis.
- If X_t is Gaussian, then the representation $\{X_n\}$ is a vector of independent, Gaussian random variables.
- For white noise, $K_X(t,u) = \frac{N_0}{2}\delta(t-u)$. Then, the eigenfunctions must satisfy

$$\int \frac{N_0}{2}\delta(t-u)\Phi(u)\,du = \lambda\Phi(t).$$

  - Any orthonormal set of bases $\{\Phi\}$ satisfies this condition!
  - Eigenvalues λ are all equal to $\frac{N_0}{2}$.
  - If X_t is white, Gaussian noise then the representation $\{X_n\}$ are independent, identically distributed random variables.
    - zero mean
    - variance $\frac{N_0}{2}$