Integrals of Random Processes

- We will see, that receivers always include a linear, time-invariant system, i.e., a filter.
- Linear, time-invariant systems *convolve* the input random process with the impulse response of the filter.
  - Convolution is fundamentally an integration.
- We will establish conditions that ensure that an expression like

$$Z(\omega) = \int_a^b X_t(\omega) h(t) \, dt$$

is "well-behaved".
  - The result of the (definite) integral is a random variable.
- **Concern:** Does the above integral *converge*?

Mean Square Convergence

- There are different senses in which a sequence of random variables may converge: *almost surely*, *in probability*, *mean square*, and *in distribution*.
- We will focus exclusively on mean square convergence.
- For our integral, mean square convergence means that the Rieman sum and the random variable Z satisfy:
  - Given $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\mathbf{E}[(\sum_{k=1}^{n} X_{\tau_k} h(\tau_k)(t_k - t_{k-1}) - Z)^2] \leq \epsilon.$$

    with:
    - $a = t_0 < t_1 < \cdots < t_n = b$
    - $t_{k-1} \leq \tau_k \leq t_k$
    - $\delta = \max_k (t_k - t_{k-1})$

Mean Square Convergence — Why We Care

- It can be shown that the integral converges if

$$\int_a^b \int_a^b R_X(t,u) h(t) h(u)\, dt\, du < \infty$$

  - We will see shortly that this implies $\mathbf{E}[|Z|^2] < \infty$.
- **Important:** When the integral converges, then the order of integration and expectation can be interchanged, e.g.,

$$\mathbf{E}[Z] = \mathbf{E}[\int_a^b X_t h(t)\, dt] = \int_a^b \mathbf{E}[X_t] h(t)\, dt = \int_a^b m_X(t) h(t)\, dt$$

- Throughout this class, we will focus exclusively on cases where $R_X(t,u)$ and $h(t)$ are such that our integrals converge.

Exercise: Brownian Motion

- **Definition:** Let N_t be white Gaussian noise with $\frac{N_0}{2} = \sigma^2$. The random process

$$W_t = \int_0^t N_s \, ds \quad \text{for } t \geq 0$$

  is called Brownian Motion or Wiener Process.

- Compute the mean and autocorrelation functions of W_t.
- **Answer:** $m_W(t) = 0$ and $R_W(t, u) = \sigma^2 \min(t, u)$

Integrals of Gaussian Random Processes

- Let X_t denote a Gaussian random process with second order description $m_X(t)$ and $R_X(t,s)$.
- Then, the integral

$$Z = \int_a^b X(t)h(t)\,dt$$

is a Gaussian random variable.
- Moreover mean and variance are given by

$$\mu = \mathbf{E}[Z] = \int_a^b m_X(t)h(t)\,dt$$

$$\begin{aligned}\mathrm{Var}[Z] &= \mathbf{E}[(Z-\mathbf{E}[Z])^2] = \mathbf{E}[(\int_a^b (X_t - m_x(t))h(t)\,dt)^2] \\ &= \int_a^b\int_a^b C_X(t,u)h(t)h(u)\,dt\,du\end{aligned}$$

Jointly Defined Random Processes

- Let X_t and Y_t be jointly defined random processes.
  - E.g., input and output of a filter.
- Then, joint densities of the form $p_{X_t Y_u}(x, y)$ can be defined.
- Additionally, second order descriptions that describe the correlation between samples of X_t and Y_t can be defined.

Crosscorrelation and Crosscovariance

- **Definition:** The crosscorrelation function $R_{XY}(t,u)$ is defined as:

$$R_{XY}(t,u) = \mathbf{E}[X_t Y_u] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp_{X_t Y_u}(x,y)\,dx\,dy.$$

- **Definition:** The crosscovariance function $C_{XY}(t,u)$ is defined as:

$$C_{XY}(t,u) = R_{XY}(t,u) - m_X(t)m_Y(u).$$

- **Definition:** The processes X_t and Y_t are called jointly wide-sense stationary if:
  1. $R_{XY}(t,u) = R_{XY}(t-u)$ **and**
  2. $m_X(t)$ and $m_Y(t)$ are constants.

Filtering of Random Processes

Filtered Random Process

$$X_t \longrightarrow \boxed{h(t)} \longrightarrow Y_t$$

Filtering of Random Processes

- Clearly, X_t and Y_t are jointly defined random processes.
- Standard LTI system — convolution:

$$Y_t = \int h(t-\sigma) X_\sigma \, d\sigma = h(t) * X_t$$

- Recall: this convolution is "well-behaved" if

$$\int\int R_X(\sigma,\nu) h(t-\sigma) h(t-\nu) \, d\sigma \, d\nu < \infty$$

  - E.g.: $\int\int R_X(\sigma,\nu) \, d\sigma \, d\nu < \infty$ **and** $h(t)$ stable.

Second Order Description of Output: Mean

- The expected value of the filter's output Y_t is:

$$\begin{aligned}\mathbf{E}[Y_t] &= \mathbf{E}[\int h(t-\sigma)X_\sigma \, d\sigma] \\ &= \int h(t-\sigma)\mathbf{E}[X_\sigma] \, d\sigma \\ &= \int h(t-\sigma)m_X(\sigma) \, d\sigma\end{aligned}$$

- For a wss process X_t, $m_X(t)$ is constant. Therefore,

$$\mathbf{E}[Y_t] = m_Y(t) = m_X \int h(\sigma) \, d\sigma$$

is also constant.

Crosscorrelation of Input and Output

- The crosscorrelation between input and ouput signals is:

$$R_{XY}(t,u) = \mathbf{E}[X_t Y_u] = \mathbf{E}[X_t \int h(u-\sigma) X_\sigma \, d\sigma$$
$$= \int h(u-\sigma) \mathbf{E}[X_t X_\sigma] \, d\sigma$$
$$= \int h(u-\sigma) R_X(t,\sigma) \, d\sigma$$

- For a wss input process

$$R_{XY}(t,u) = \int h(u-\sigma) R_X(t,\sigma) \, d\sigma = \int h(\nu) R_X(t, u-\nu) \, d\nu$$
$$= \int h(\nu) R_X(t-u+\nu) \, d\nu = R_{XY}(t-u)$$

  - Input and output are jointly stationary.

Autocorelation of Output

- The autocorrelation of Y_t is given by

$$R_Y(t,u) = \mathbf{E}[Y_t Y_u] = \mathbf{E}[\int h(t-\sigma) X_\sigma \, d\sigma \int h(u-\nu) X_\nu \, d\nu]$$
$$= \iint h(t-\sigma) h(u-\nu) R_X(\sigma,\nu) \, d\sigma \, d\nu$$

- For a wss input process:

$$R_Y(t,u) = \iint h(t-\sigma) h(u-\nu) R_X(\sigma,\nu) \, d\sigma \, d\nu$$
$$= \iint h(\lambda) h(\lambda-\gamma) R_X(t-\lambda, u-\lambda+\gamma) \, d\lambda \, d\gamma$$
$$= \iint h(\lambda) h(\lambda-\gamma) R_X(t-u-\gamma) \, d\lambda \, d\gamma = R_Y(t-u)$$

  - Define $R_h(\gamma) = \int h(\lambda) h(\lambda-\gamma) \, d\lambda = h(\lambda) * h(-\lambda)$.
  - Then, $R_Y(\tau) = \int R_h(\gamma) R_X(\tau-\gamma) \, d\gamma = R_h(\tau) * R_X(\tau)$

Exercise: Filtered White Noise Process

- Let the white Gaussian noise process X_t be input to a filter with impulse response

$$h(t) = e^{-at}u(t) = \begin{cases} e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

- Compute the second order description of the output process Y_t.
- **Answers:**
  - Mean: $m_Y = 0$
  - Autocorrelation:

$$R_Y(\tau) = \frac{N_0}{2} \frac{e^{-a|\tau|}}{2a}$$

Power Spectral Density — Concept

- **Power Spectral Density** (PSD) measures how the power of a random process is distributed over frequency.
  - Notation: $S_X(f)$
  - Units: Watts per Hertz (W/Hz)
- Thought experiment:
  - Pass random process X_t through a narrow bandpass filter:
    - center frequency f
    - bandwidth Δf
    - denote filter output as Y_t
  - Measure the power P at the output of bandpass filter:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |Y_t|^2 \, dt$$

  - Relationship between power and (PSD)

$$P \approx S_X(f) \cdot \Delta f.$$

Relation to Autocorrelation Function

- For a wss random process, the power spectral density is closely related to the autocorrelation function $R_X(\tau)$.
- **Definition:** For a random process X_t with autocorrelation function $R_X(\tau)$, the power spectral density $S_X(f)$ is defined as the Fourier transform of the autocorrelation function,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{j2\pi f\tau}\, d\tau.$$

  - For non-stationary processes, it is possible to define a spectral represenattion of the process.
  - However, the spectral contents of a non-stationary process will be time-varying.
- **Example:** If N_t is white noise, i.e., $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$, then

$$S_X(f) = \frac{N_0}{2} \quad \text{for all } f$$

Properties of the PSD

- Inverse Transform:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{-j2\pi f\tau}\, df.$$

- The total power of the process is

$$\mathbf{E}[|X_t|^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df.$$

- $S_X(f)$ is even and non-negative.
  - Evenness of $S_X(f)$ follows from evenness of $R_X(\tau)$.
  - Non-negativeness is a consequence of the autocorrelation function being positive definite

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) f^*(u) R_X(t,u)\, dt\, du \geq 0$$

for all choices of $f(\cdot)$, including $f(t) = e^{-j2\pi ft}$.

Filtering of Random Processes

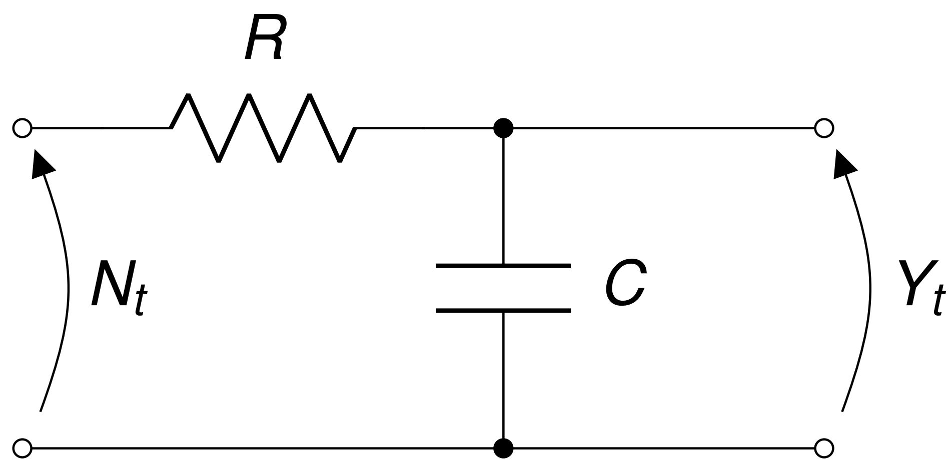
- Random process X_t with autocorrelation $R_X(\tau)$ and PSD $S_X(f)$ is input to LTI filter with impuse response $h(t)$ and frequency response $H(f)$.
- The PSD of the output process Y_t is

$$S_Y(f) = |H(f)|^2 S_X(f).$$

  - Recall that $R_Y(\tau) = R_X(\tau) * C_h(\tau)$,
  - where $C_h(\tau) = h(\tau) * h(-\tau)$.
  - In frequency domain: $S_Y(f) = S_X(f) \cdot \mathcal{F}\{C_h(\tau)\}$
  - With

$$\begin{aligned}
\mathcal{F}\{C_h(\tau)\} &= \mathcal{F}\{h(\tau) * h(-\tau)\} \\
&= \mathcal{F}\{h(\tau)\} \cdot \mathcal{F}\{h(-\tau)\} \\
&= H(f) \cdot H^*(f) = |H(f)|^2.
\end{aligned}$$

Exercise: Filtered White Noise



- Let N_t be a white noise process that is input to the above circuit. Find the power spectral density of the output process.
- **Answer:**

$$S_Y(f) = \left| \frac{1}{1 + j2\pi fRC} \right|^2 \frac{N_0}{2} = \frac{1}{1 + (2\pi fRC)^2} \frac{N_0}{2}.$$