Random Processes — Why we Care

- Random processes describe signals that change randomly over time.
  - Compare: deterministic signals can be described by a mathematical expression that describes the signal exactly for all time.
  - Example: $x(t) = 3\cos(2\pi f_c t + \pi/4)$ with $f_c = 1$GHz.
- We will encounter three types of random processes in communication systems:
  1. (nearly) deterministic signal with a random parameter — Example: sinusoid with random phase.
  2. signals constructed from a sequence of random variables — Example: digitally modulated signals with random symbols
  3. noise-like signals
- **Objective:** Develop a framework to describe and analyze random signals encountered in the receiver of a communication system.

Random Process — Formal Definition

- Random processes can be defined completely analogous to random variables over a probability triple space $(\Omega, \mathcal{F}, P)$.
- **Definition:** A random process is a mapping from each element ω of the sample space Ω to a function of time (i.e., a signal).
- Notation: $X_t(\omega)$ — we will frequently omit ω to simplify notation.
- Observations:
  - We will be interested in both real and complex valued random processes.
  - Note, for a given random outcome ω_0, $X_t(\omega_0)$ is a *deterministic* signal.
  - Note, for a fixed time t_0, $X_{t_0}(\omega)$ is a *random variable*.

Sample Functions and Ensemble

- For a given random outcome ω_0, $X_t(\omega_0)$ is a deterministic signal.
  - Each signal that that can be produced by a our random process is called a sample function of the random process.
- The collection of all sample functions of a random process is called the ensemble of the process.
- **Example:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
  The ensemble of this random process consists of the four sample functions:

$$X_t(\omega_1) = \cos(2\pi f_0 t) \qquad X_t(\omega_2) = -\sin(2\pi f_0 t)$$
$$X_t(\omega_3) = -\cos(2\pi f_0 t) \qquad X_t(\omega_4) = \sin(2\pi f_0 t)$$

Probability Distribution of a Random Process

- For a given time instant t, $X_t(\omega)$ is a random variable.
- Since it is a random variable, it has a pdf (or pmf in the discrete case).
  - We denote this pdf as $p_{X_t}(x)$.
- The statistical properties of a random process are specified completely if the joint pdf

$$p_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n)$$

  is available for all n and t_i, $i = 1, \ldots, n$.
  - This much information is often not available.
  - Joint pdfs with many sampling instances can be cumbersome.
  - We will shortly see a more concise summary of the statistics for a random process.

Random Process with Random Parameters

- A deterministic signal that depends on a random parameter is a random process.
  - Note, the sample functions of such random processes do not "look" random.
- Running Examples:
  - **Example (discrete phase):** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
  - **Example (continuous phase):** same as above but phase $\Theta(\omega)$ is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim U[0, 2\pi)$.
- For both of these processes, the complete statistical description of the random process can be found.

Example: Discrete Phase Process

- **Discrete Phase Process:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- Find the first-order density $p_{X_t}(x)$ for this process.
- Find the second-order density $p_{X_{t_1} X_{t_2}}(x_1, x_2)$ for this process.
  - Note, since the phase values are discrete the above pdfs must be expressed with the help of δ-functions.
  - Alternatively, one can derive a probability mass function.

Solution: Discrete Phase Process

- First-order density function:

$$p_{X_t}(x) = \frac{1}{4}(\delta(x - \cos(2\pi f_0 t)) + \delta(x + \sin(2\pi f_0 t)) + \delta(x + \cos(2\pi f_0 t)) + \delta(x - \sin(2\pi f_0 t)))$$

- Second-order density function:

$$\begin{aligned} p_{X_{t_1} X_{t_2}}(x_1, x_2) = \frac{1}{4}(&\delta(x_1 - \cos(2\pi f_0 t_1)) \cdot \delta(x_2 - \cos(2\pi f_0 t_2)) + \\ &\delta(x_1 + \sin(2\pi f_0 t_1)) \cdot \delta(x_2 + \sin(2\pi f_0 t_2)) + \\ &\delta(x_1 + \cos(2\pi f_0 t_1)) \cdot \delta(x_2 + \cos(2\pi f_0 t_2)) + \\ &\delta(x_1 - \sin(2\pi f_0 t_1)) \cdot \delta(x_2 - \sin(2\pi f_0 t_2))) \end{aligned}$$

Example: Continuous Phase Process

- **Continuous Phase Process:** Let $\Theta(\omega)$ be a random variable that is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim [0, 2\pi)$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- Find the first-order density $p_{X_t}(x)$ for this process.
- Find the second-order density $p_{X_{t_1} X_{t_2}}(x_1, x_2)$ for this process.

Solution: Continuous Phase Process

- First-order density:

$$p_{X_t}(x) = \frac{1}{\pi\sqrt{1-x^2}} \qquad \text{for } |x| \leq 1.$$

  Notice that $p_{X_t}(x)$ does **not** depend on t.

- Second-order density:

$$p_{X_{t_1}X_{t_2}}(x_1, x_2) = \frac{1}{\pi\sqrt{1-x_2^2}} \cdot [\frac{1}{2} \cdot$$
$$\delta(x_1 - \cos(2\pi f_0(t_1 - t_2) + \arccos(x_2))) +$$
$$\delta(x_1 - \cos(2\pi f_0(t_1 - t_2) - \arccos(x_2)))]$$

Random Processes Constructed from Sequence of Random Experiments

- Model for digitally modulated signals.
- Example:
  - Let $X_k(\omega)$ denote the outcome of the k-th toss of a coin:

$$X_k(\omega) = \begin{cases} 1 & \text{if heads on } k\text{-th toss} \\ -1 & \text{if tails on } k\text{-th toss.} \end{cases}$$

  - Let $p(t)$ denote a pulse of duration T, e.g.,

$$p(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T \\ 0 & \text{else.} \end{cases}$$

  - Define the random process X_t

$$X_t(\omega) = \sum_k X_k(\omega) p(t - nT)$$

Probability Distribution

- Assume that heads and tails are equally likely.
- Then the first-order density for the above random process is

$$p_{X_t}(x) = \frac{1}{2}(\delta(x-1) + \delta(x+1)).$$

- The second-order density is:

$$p_{X_{t_1} X_{t_2}}(x_1, x_2) = \begin{cases} \delta(x_1 - x_2) p_{X_{t_1}}(x_1) & \text{if } nT \leq t_1, t_2 \leq (n+1)T \\ p_{X_{t_1}}(x_1) p_{X_{t_2}}(x_2) & \text{else.} \end{cases}$$

  - These expression become more complicated when $p(t)$ is not a rectangular pulse.

Probability Density of Random Processs Defined Directly

- Sometimes the n-th order probability distribution of the random process is given.
  - Most important example: Gaussian Random Process
    - Statistical model for noise.
  - **Definition:** The random process X_t is Gaussian if the vector $\vec{X}$ of samples taken at times $t_1, \ldots, t_n$

$$\vec{X} = \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}$$

is a Gaussian random vector for all $t_1, \ldots, t_n$.

Second Order Description of Random Processes

- Characterization of random processes in terms of n-th order densities is
  - frequently not available
  - mathematically cumbersome
- A more tractable, practical alternative description is provided by the second order description for a random process.
- **Definition:** The second order description of a random process consists of the
  - mean function and the
  - autocorrelation function

  of the process.
- Note, the second order description can be computed from the (second-order) joint density.
  - The converse is not true — at a minimum the distribution must be specified (e.g., Gaussian).

Mean Function

- The second order description of a process relies on the mean and autocorrelation functions — these are defined as follows
- **Definition:** The mean of a random process is defined as:

$$\mathbf{E}[X_t] = m_X(t) = \int_{-\infty}^{\infty} x \cdot p_{X_t}(x)\, dx$$

  - Note, that the mean of a random process is a deterministic signal.
  - The mean is computed from the first oder density function.

Autocorrelation Function

- **Definition:** The autocorrelation function of a random process is defined as:

$$R_X(t,u) = \mathbf{E}[X_t X_u] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p_{X_t,X_u}(x,y)\,dx\,dy$$

  - Autocorrelation is computed from second order density

Autocovariance Function

- Closely related: autocovariance function:

$$\begin{aligned} C_X(t,u) &= \mathbf{E}[(X_t - m_X(t))(X_u - m_X(u))] \\ &= R_X(t,u) - m_X(t)m_X(u) \end{aligned}$$

Exercise: Discrete Phase Example

- Find the second-order description for the discrete phase random process.
  - **Discrete Phase Process:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- **Answer:**
  - Mean: $m_X(t) = 0$.
  - Autocorrelation function:

$$R_X(t, u) = \frac{1}{2} \cos(2\pi f_0 (t - u)).$$

Exercise: Continuous Phase Example

- Find the second-order description for the continuous phase random process.
  - **Continuous Phase Process:** Let $\Theta(\omega)$ be a random variable that is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim [0, 2\pi)$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- **Answer:**
  - Mean: $m_X(t) = 0$.
  - Autocorrelation function:

$$R_X(t, u) = \frac{1}{2} \cos(2\pi f_0 (t - u)).$$

Properties of the Autocorrelation Function

- The autocorrelation function of a (real-valued) random process satisfies the following properties:
  1. $R_X(t,t) \geq 0$
  2. $R_X(t,u) = R_X(u,t)$ (symmetry)
  3. $|R_X(t,u)| \leq \frac{1}{2}(R_X(t,t) + R_X(u,u))$
  4. $|R_X(t,u)|^2 \leq R_X(t,t) \cdot R_X(u,u)$

Stationarity

- The concept of stationarity is analogous to the idea of time-invariance in linear systems.
- **Interpretation:** For a stationary random process, the statistical properties of the process do not change with time.
- **Definition:** A random process X_t is strict-sense stationary (sss) to the n-th order if:

$$p_{X_{t_1},\ldots,X_{t_n}}(x_1,\ldots,x_n) = p_{X_{t_1+T},\ldots,X_{t_n+T}}(x_1,\ldots,x_n)$$

  for all T.
  - The statistics of X_t do not depend on *absolute* time but only on the time differences between the sample times.

Wide-Sense Stationarity

- A simpler and more tractable notion of stationarity is based on the second-order description of a process.
- **Definition:** A random process X_t is wide-sense stationary (wss) if
  1. the mean function $m_X(t)$ is constant **and**
  2. the autocorrelation function $R_X(t, u)$ depends on t and u only through $t-u$, i.e., $R_X(t, u) = R_X(t-u)$
- **Notation:** for a wss random process, we write the autocorrelation function in terms of the single time-parameter $\tau = t - u$:

$$R_X(t, u) = R_X(t-u) = R_X(\tau).$$

Exercise: Stationarity

- **True or False:** Every random process that is strict-sense stationarity to the second order is also wide-sense stationary.
  - **Answer:** True
- **True or False:** Every random process that is wide-sense stationary must be strict-sense stationarity to the second order.
  - **Answer:** False
- **True or False:** The discrete phase process is strict-sense stationary.
  - **Answer:** False; first order density depends on t, therefore, not even first-order sss.
- **True or False:** The discrete phase process is wide-sense stationary.
  - **Answer:** True

White Gaussian Noise

- **Definition:** A (real-valued) random process X_t is called white Gaussian Noise if
  - X_t is Gaussian for each time instance t
  - Mean: $m_X(t) = 0$ for all t
  - Autocorrelation function: $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$
  - White Gaussian noise is a good model for noise in communication systems.
  - Note, that the variance of X_t is infinite:

$$\mathrm{Var}(X_t) = \mathbf{E}[X_t^2] = R_X(0) = \frac{N_0}{2}\delta(0) = \infty.$$

  - Also, for $t \neq u$: $\mathbf{E}[X_t X_u] = R_X(t, u) = R_X(t - u) = 0$.