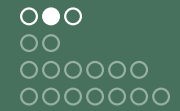




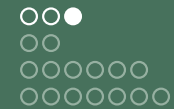
Introduction

- ▶ We compare methods for transmitting a sequence of bits.
- ▶ We will see that the performance of these methods varies significantly.
- ▶ Main Result: It is possible to achieve error free communications as long as SNR is good enough and data rate is not too high.



Problem Statement

- ▶ **Problem:**
 - ▶ K bits must be transmitted in T seconds.
 - ▶ Available power is limited to P .
- ▶ **Question:** What method achieves the lowest probability of error?



Parameters

► Data Rate:

$$R = \frac{\log_2 M}{T} \quad (\text{bits/s})$$

- T = symbol (baud) rate
- M = alphabet size
- implicit assumption: symbols are equally likely.

► Power and energy:

$$P = \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{E_s}{T}$$

- $s(t)$ is of duration T .
- Note, bit energy is given by

$$E_b = \frac{E_s}{\log_2 M} = \frac{PT}{\log_2 M} = \frac{P}{R}$$



Bit-by-bit Signaling

- ▶ Transmit K bit as a sequence of “one-shot” BPSK signals.
- ▶ $K = RT$ bits to be transmitted.
- ▶ Energy per bit E_b .
- ▶ Consider, signals of the form

$$s(t) = \sum_{k=0}^{K-1} \sqrt{E_b} s_k p(t - k/R)$$

- ▶ $s_k \in \{\pm 1\}$
- ▶ $p(t)$ is a pulse of duration $1/R = T/K$ and $\|p(t)\|^2 = 1$.
- ▶ **Question:** What is the probability that any transmission error occurs?
 - ▶ In other words, the transmission is not received without error.

Error Probability for Bit-by-Bit Signaling

- ▶ We can consider the entire set of transmissions as a K -dimensional signal set.
 - ▶ Signals are at the vertices of a K -dimensional hypercube.

$$\begin{aligned}\Pr\{\mathbf{e}\} &= 1 - \left(1 - Q\left(\frac{2E_b}{N_0}\right)\right)^K \\ &= 1 - \left(1 - Q\left(\frac{2P}{RN_0}\right)\right)^{RT}\end{aligned}$$

- ▶ Note, for any E_b/N_0 and R , the error rate will always tend to 1 as $T \rightarrow \infty$.
 - ▶ Error-free transmission is *not* possible with bit-by-bit signaling.



Block-Orthogonal Signaling

- ▶ Again,
 - ▶ $K = RT$ bits are transmitted in T seconds.
 - ▶ Energy per bit E_b .
- ▶ Signal set (Pulse-position modulation — PPM)

$$s_k(t) = \sqrt{E_s} p(t - kT/2^K) \quad \text{for } k = 0, 1, \dots, 2^K - 1.$$

where $p(t)$ is of duration $T/2^K$ and $\|p(t)\|^2 = 1$.

- ▶ Alternative signal set (Frequency Shift Keying — FSK)

$$s_k(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi(f_c + k/T)t) \quad \text{for } k = 0, 1, \dots, 2^K - 1.$$

- ▶ Since the signal set consists of $M = 2^K$ signals, each signal conveys K bits — each signal occupies one dimension.
- ▶ Note that $E_s = KE_b$.

Union Bound

- ▶ The error probability for block-orthogonal signaling cannot be computed in closed form.
- ▶ At high and moderate SNR, the error probability is well approximated by the union bound.
 - ▶ Each signal has $M - 1 = 2^K - 1$ nearest neighbors.
 - ▶ The distance between neighbors is $d_{\min} = \sqrt{2E_s} = \sqrt{2KE_b}$.
- ▶ Union bound

$$\begin{aligned} \Pr\{e\} &\leq (2^K - 1) Q \left(\sqrt{\frac{KE_b}{N_0}} \right) \\ &= (2^{RT} - 1) Q \left(\sqrt{\frac{PT}{N_0}} \right) \end{aligned}$$



Bounding the Union Bound

- ▶ To gain further insight, we bound

$$Q(x) \leq \frac{1}{2} \exp(-x^2/2).$$

- ▶ Then,

$$\begin{aligned} \Pr\{e\} &\leq (2^{RT} - 1) Q\left(\sqrt{\frac{PT}{N_0}}\right) \\ &\lesssim 2^{RT} \exp\left(-\frac{PT}{2N_0}\right) \\ &= \exp\left(-T\left(\frac{P}{2N_0} - R \ln 2\right)\right). \end{aligned}$$

- ▶ Hence, $\Pr\{e\} \rightarrow 0$ as $T \rightarrow \infty$!
 - ▶ As long as $R < \frac{1}{\ln 2} \frac{P}{2N_0}$.

- ▶ **Error-free transmission is possible!**



Reality-Check: Bandwidth

- ▶ **Bit-by-bit Signaling:** Pulse-width: $T/K = 1/R$.
 - ▶ Bandwidth is approximately equal to $B = R$.
 - ▶ Also, number of dimensions $K = RT$.
- ▶ **Block-orthogonal:** Pulse width: $T/2^K = T/2^{RT}$.
 - ▶ Bandwidth is approximately equal to $B = 2^{RT}/T$.
 - ▶ Number of dimensions is $2^K = 2^{RT}$.
- ▶ Bandwidth for block-orthogonal signaling grows exponentially with the number of bits K .
 - ▶ Not practical for moderate to large blocks of bits.



The Dimensionality Theorem

- ▶ The relationship between bandwidth B and the number of dimensions is summarized by the *dimensionality theorem*:
 - ▶ The number of dimensions D available over an interval of duration T is limited by the bandwidth B

$$D \leq B \cdot T$$

- ▶ The theorem implies:
 - ▶ A signal occupying D dimensions over T seconds requires bandwidth

$$B \geq \frac{D}{T}$$



An Ideal Signal Set

- ▶ An ideal signal set combines the aspects of our two example signal sets:
 - ▶ $\Pr\{e\}$ -behavior like block orthogonal signaling

$$\lim_{T \rightarrow \infty} \Pr\{e\} = 0.$$

- ▶ Bandwidth behavior like bit-by-bit signaling

$$B = \frac{D}{T} = \text{constant}.$$

- ▶ Thus, $D = BT \Rightarrow \infty$ as $T \rightarrow \infty$.
 - ▶ **Question:** Does such a signal set exist?



Towards Channel Capacity

▶ Given:

- ▶ bandwidth $B = \frac{D}{T}$, where T is the duration of the transmission.
- ▶ power P
- ▶ Noise power spectral density $\frac{N_0}{2}$

▶ Question: What is the highest data rate R that allows error-free transmission with the above constraints?

- ▶ We are transmitting RT bits
- ▶ Therefore, we need $M = 2^{RT}$ signals.

Signal Set

- ▶ Our signal set consists of $M = 2^{RT}$ signals of the form

$$s_n(t) = \sum_{k=0}^{D-1} X_{n,k} p(t - kT/D)$$

where

- ▶ $p(t)$ are pulses of duration T/D , i.e., of bandwidth $B = D/T$.
- ▶ Also, $\|p(t)\|^2 = 1$.
- ▶ Each signal $s_n(t)$ is defined by a vector $\vec{X}_n = \{X\}_{n,k}$.
- ▶ We are looking to find $M = 2^{RT}$ length- D vectors \vec{X} that lead to good error properties.
- ▶ Note that the signals $p(t - kT/D)$ form an orthonormal basis.

Receiver Frontend

- ▶ The receiver frontend consists of a matched filter for $p(t)$ followed by a sampler at times kT/D .
 - ▶ I.e., the frontend projects the received signal onto the orthonormal basis functions $p(t - kT/D)$.

- ▶ The vector \vec{R} of matched filter outputs has elements

$$R_k = \langle R_t, p(t - kT/D) \rangle \quad k = 0, 1, \dots, D - 1$$

- ▶ Conditional on $s_k(t)$ was sent, $\vec{R} \sim N(\vec{X}_k, \frac{N_0}{2} I)$.
- ▶ The optimum receiver selects that signal that's closest to \vec{R} .

Conditional Error Probability

- ▶ When, the signal $s_k(t)$ was sent then $\vec{R} \sim N(\vec{X}_k, \frac{N_0}{2} I)$.
- ▶ As the number of dimensions D increases, the vector \vec{R} lies within a sphere with center \vec{X}_k and radius $\sqrt{D \frac{N_0}{2}}$ with very high probability $(1 - e^{-D})$.
 - ▶ **Important:** We allow the radius of the decoding spheres to grow with the number of dimensions D .
 - ▶ This ensures that $P_e \rightarrow 0$ as $D \rightarrow \infty$.
- ▶ We call the spheres of radius $\sqrt{D \frac{N_0}{2}}$ around each signal point *decoding spheres*.
 - ▶ The decoding spheres will be part of the decision regions for each point.

Power Constraint

- ▶ The signal power must satisfy

$$\frac{1}{T} \int_0^T s^2(t) dt = \frac{1}{T} \sum_{k=0}^{D-1} |X_k|^2 \leq P.$$

- ▶ Therefore,

$$\sum_{k=0}^{D-1} |X_k|^2 \leq PT.$$

- ▶ Insight: The observed signals must lie in a large sphere of radius $\sqrt{PT + D\frac{N_0}{2}}$.
- ▶ **Question:** How many decoding spheres can we have and still meet the power constraint?



Capacity

- ▶ Each decoding sphere has volume $K_D \left(\sqrt{D \frac{N_0}{2}} \right)^D$.
- ▶ The volume of the sphere containing the observed signals is $K_D \left(\sqrt{PT + D \frac{N_0}{2}} \right)^D$
 - ▶ K_D is a constant that depends only on the number of dimensions D .
- ▶ The number of decoding spheres that fit into the the power sphere is (upper) bounded by the ratio of the volumes

$$\frac{K_D \left(\sqrt{PT + D \frac{N_0}{2}} \right)^D}{K_D \left(\sqrt{D \frac{N_0}{2}} \right)^D}$$



Capacity

- ▶ Since the number of signals $M = 2^{RT}$ equals the number of decoding spheres, it follows that error free communications is possible (in the limit as $D \rightarrow \infty$) if

$$M = 2^{RT} \leq \frac{\left(\sqrt{PT + D\frac{N_0}{2}}\right)^D}{\left(\sqrt{D\frac{N_0}{2}}\right)^D}$$

or

$$R < \frac{D}{2T} \log_2\left(1 + \frac{PT}{DN_0/2}\right) = \frac{B}{2} \log_2\left(1 + \frac{P}{BN_0/2}\right).$$

- ▶ Note, if we allow *complex valued* signals, then $R < B \log_2\left(1 + \frac{P}{BN_0}\right)$.