Exercise: QPSK

- Find the error rate for the signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3.$$

- **Answer:** (Recall $\eta_P = \frac{d^2_{\min}}{E_b} = 4$ for QPSK)

$$\begin{aligned} \Pr\{e\} &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right) \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ &= 2Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right). \end{aligned}$$

Exercise: 16-QAM

(Recall $\eta_P = \frac{d^2_{\min}}{E_b} = \frac{8}{5}$ for 16-QAM)

- Find the error rate for the signal set $(a_I, a_Q \in \{-3, -1, 1, 3\})$

$$s_n(t) = \sqrt{2E_0/T} a_I \cdot \cos(2\pi f_c t) + \sqrt{2E_0/T} a_Q \cdot \sin(2\pi f_c t)$$

- **Answer:** $(\eta_P = \frac{d^2_{\min}}{E_b} = 4)$

$$\begin{aligned}
\Pr\{e\} &= 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{2E_0}{N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$

N-dimensional Hypercube

- Find the error rate for the signal set with 2^N signals of the form ($b_{k,n} \in \{-1, 1\}$):

$$s_n(t) = \sum_{k=1}^{N} \sqrt{\frac{2E_s}{NT}} b_{k,n} \cos(2\pi nt/T), \text{ for } 0 \leq t \leq T$$

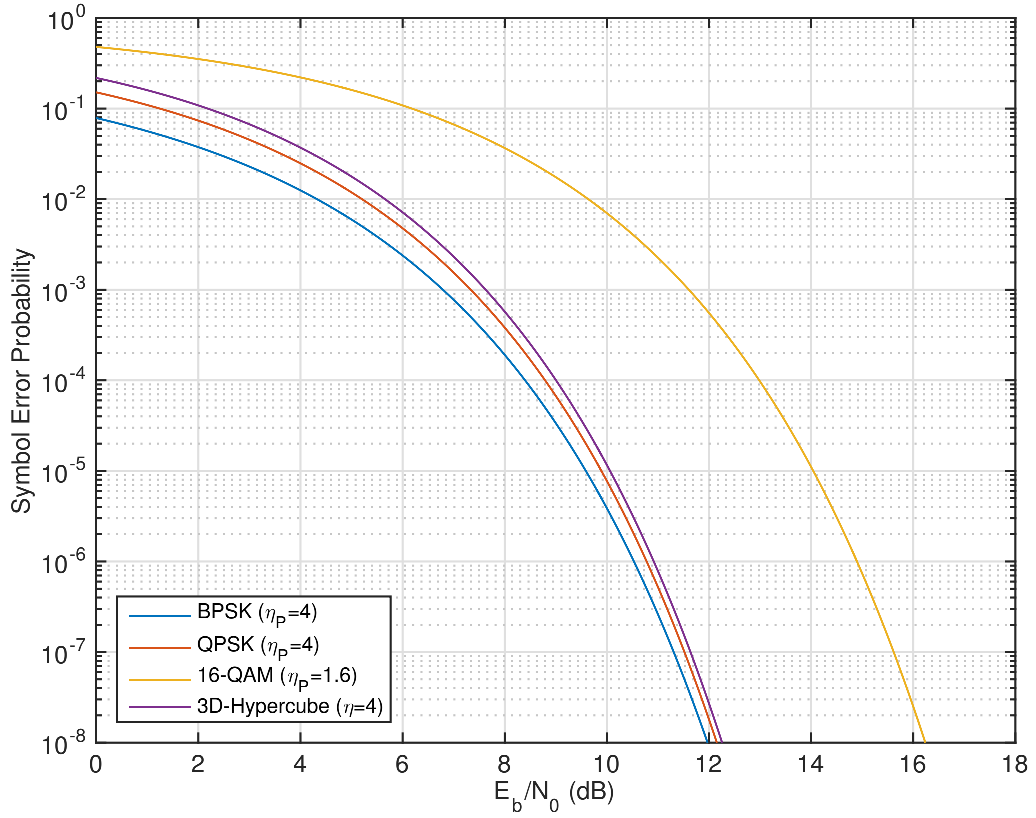
- **Answer:**

$$\Pr\{e\} = 1 - \left(1 - Q\left(\sqrt{\frac{2E_s}{N \cdot N_0}}\right)\right)^N$$

$$= 1 - \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^N$$

$$= 1 - \left(1 - Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right)\right)^N \approx N \cdot Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).$$

Comparison



- Better power efficiency η_P leads to better error performance (at high SNR).

What if Decision Regions are not Rectangular?

- **Example:** For 8-PSK, the probability of a correct decision is given by the following integral over the decision region for $s_0(t)$

$$\Pr\{c\} = \int_0^\infty \frac{1}{\sqrt{2\pi N_0/2}} \exp(-\frac{(x-\sqrt{E_s})^2}{2N_o/2} \underbrace{\int_{-x\tan(\pi/8)}^{x\tan(\pi/8)} \frac{1}{\sqrt{2\pi N_0/2}} \exp(-\frac{y^2}{2N_0/2})\, dy\, dx}_{=1-2Q(\frac{x\tan(\pi/8)}{\sqrt{N_0/2}})}$$

- This integral cannot be computed in closed form.

Union Bound

- When decision boundaries do not intersect at right angles, then the error probability cannot be computed in closed form.
- An upper bound on the conditional probability of error (assuming that s_n was sent) is provided by:

$$\begin{aligned}\Pr\{e|s_n\} &\leq \sum_{k\neq n} \Pr\{\|\vec{R}-\vec{s}_k\| < \|\vec{R}-\vec{s}_n\| | \vec{s}_n\} \\ &= \sum_{k\neq n} Q\left(\frac{\|\vec{s}_k-\vec{s}_n\|}{2\sqrt{N_0/2}}\right).\end{aligned}$$

  - Note that this bound is computed from *pairwise error probabilities* between s_n and all other signals.

Union Bound

- Then, the average probability of error can be bounded by

$$\Pr\{e\} = \sum_n \pi_n \sum_{k \neq n} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{\sqrt{2N_0}}\right).$$

- This bound is called the union bound; it approximates the union of all possible error events by the sum of the pairwise error probabilities.

Example: QPSK

- For the QPSK signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3$$

the union bound is

$$\Pr\{e\} \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

  - Recall that the exact probability of error is

$$\Pr\{e\} = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right).$$

"Intelligent" Union Bound

- The union bound is easily tightened by recognizing that only immediate neighbors of s_n must be included in the bound on the conditional error probability.
- Define the the neighbor set $N_{ML}(s_n)$ of s_n as the set of signals s_k that share a decision boundary with signal s_n.
- Then, the conditional error probability is bounded by

$$
\begin{aligned}
\Pr\{e|s_n\} &\leq \sum_{k \in N_{ML}(s_n)} \Pr\{\|\vec{R} - \vec{s}_k\| < \|\vec{R} - \vec{s}_n\| | \vec{s}_n\} \\
&= \sum_{k \in N_{ML}(s_n)} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{2\sqrt{N_0/2}}\right).
\end{aligned}
$$

"Intelligent" Union Bound

- Then, the average probability of error can be bounded by

$$\Pr\{e\} \leq \sum_n \pi_n \sum_{k \in N_{ML}(s_n)} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{\sqrt{2N_0}}\right).$$

- We refer to this bound as the intelligent union bound.
  - It still relies on pairwise error probabilities.
  - It excludes many terms in the union bound; thus, it is tighter.

Example: QPSK

- For the QPSK signal set

  $$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3$$

  the intelligent union bound includes only the immediate neighbors of each signal:

  $$\Pr\{e\} \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right).$$

  - Recall that the exact probability of error is

  $$\Pr\{e\} = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right).$$

Example: 16-QAM

- For the 16-QAM signal set, there are
  - 4 signals s_i that share a decision boundary with 4 neighbors; bound on conditional error probability: $\Pr\{e|s_i\} = 4Q(\sqrt{\frac{2E_0}{N_0}})$.
  - 8 signals s_c that share a decision boundary with 3 neighbors; bound on conditional error probability: $\Pr\{e|s_c\} = 3Q(\sqrt{\frac{2E_0}{N_0}})$.
  - 4 signals s_o that share a decision boundary with 2 neighbors; bound on conditional error probability: $\Pr\{e|s_o\} = 2Q(\sqrt{\frac{2E_0}{N_0}})$.
- The resulting intelligent union bound is

$$\Pr\{e\} \leq 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right).$$

Example: 16-QAM

- The resulting intelligent union bound is

$$\Pr\{e\} \leq 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right).$$

  - Recall that the exact probability of error is

$$\Pr\{e\} = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right).$$

Nearest Neighbor Approximation

- At high SNR, the error probability is dominated by terms that involve the shortest distance $d_{\min}$ between any pair of nodes.
  - The corresponding error probability is proportional to $Q(\sqrt{\frac{d_{\min}}{2N_0}})$.
- For each signal s_n, we count the number N_n of neighbors at distance $d_{\min}$.
- Then, the error probability at high SNR can be approximated as

$$\Pr\{e\} \approx \frac{1}{M}\sum_{n=0}^{M-1} N_n Q(\sqrt{\frac{d_{\min}^2}{2N_0}}) = \bar{N}_{\min} Q(\sqrt{\frac{d_{\min}^2}{2N_0}}).$$

Example: 16-QAM

- In 16-QAM, the distance between adjacent signals is $d_{\min} = 2\sqrt{E_0}$; also, $E_b = \frac{5}{2}E_0$.
- There are:
  - 4 signals with 4 nearest neighbors
  - 8 signals with 3 nearest neighbors
  - 4 signals with 2 nearest neighbors
- The average number of neighbors is $\bar{N}_{\min} = 3$.
- The error probability is approximately,

$$\Pr\{e\} \approx 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right).$$

  - same as the intelligent union bound.

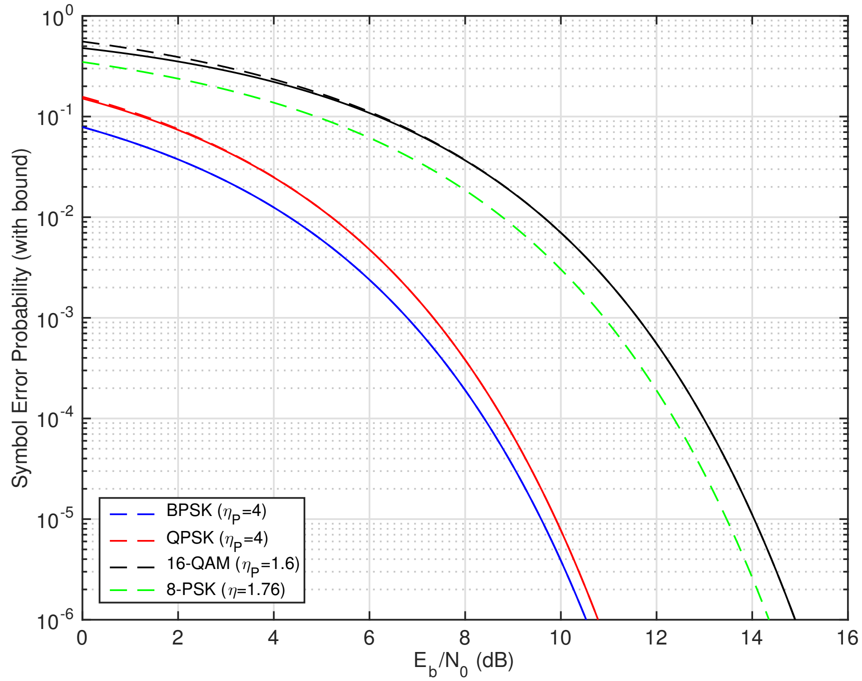
Example: 8-PSK

- For 8-PSK, each signal has 2 nearest neighbors at distance $d_{\text{min}} = \sqrt{(2-\sqrt{2})E_s}$; also, $E_b = \frac{E_s}{3}$.
- Hence, both the intelligent union bound and the nearest neighbor approximation yield

$$\Pr\{e\} \approx 2Q\left(\sqrt{\frac{(2-\sqrt{2})E_s}{2N_0}}\right) = 2Q\left(\sqrt{\frac{3(2-\sqrt{2})E_b}{2N_0}}\right)$$

  - Since, $E_b = 3E_s$.

Comparison



Solid: exact P_e, dashed: approximation. For 8PSK, only approximation is shown.

- The intelligent union bound is very tight for all cases considered here.
  - It also coincides with the nearest neighbor approximation

General Approximation for Probability of Symbol Error

- From the above examples, we can conclude that a good, general approximation for the probability of error is given by

$$\Pr\{e\} \approx \bar{N}_{\min} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = \bar{N}_{\min} Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).$$

- Probability of error depends on
  - signal-to-noise ratio (SNR) E_b/N_0 and
  - geometry of the signal constellation via the average number of neighbors $\bar{N}_{\min}$ and the power efficiency η_P.