More than Two Signals: M-ary Signal Sets

Q: How do you extend receiver concepts to signal set with $M > 2$ signals

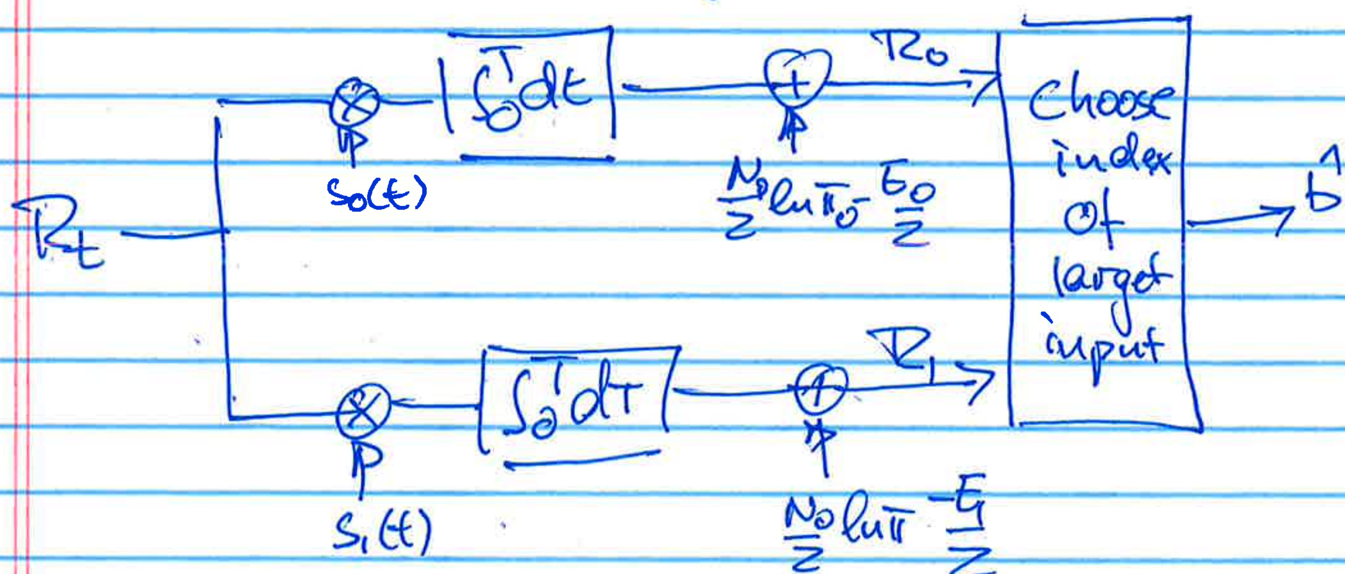
Start with optimum receiver:

$$\int_0^T R_t \cdot (s_0(t) - s_1(t))\,dt \underset{\hat{b}=1}{\overset{\hat{b}=0}{\gtrless}} \frac{N_0}{2} \ln \frac{\pi_1}{\pi_0} + \frac{1}{2}(E_0 - E_1)$$

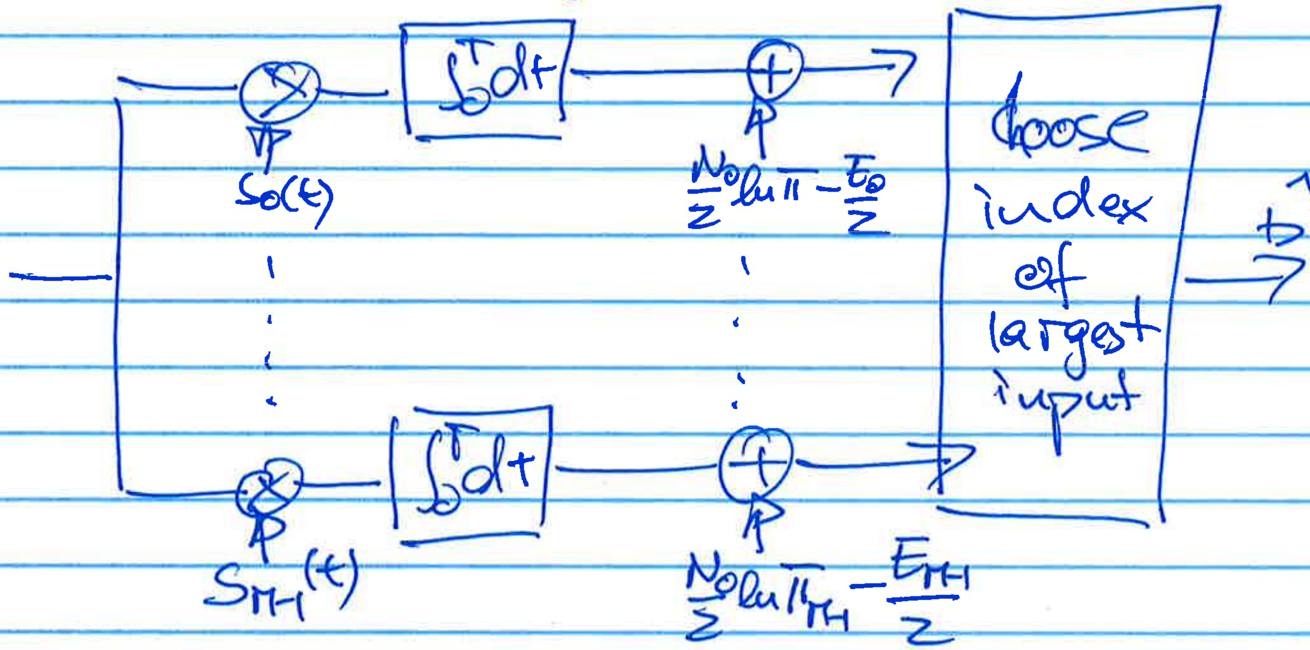
This is equivalen to:

$$\int_0^T R(t) \cdot s_0(t)\,dt + \frac{N_0}{2} \ln \pi_0 - \frac{E_0}{2} \underset{\hat{b}=1}{\overset{\hat{b}=0}{\gtrless}} \int_0^T R(t)\, s_1(t)\,dt + \frac{N_0}{2} \ln \pi_1 - \frac{E_1}{2}$$

As a block diagram



This structure is easily extended to $M > 2$ signals:



<u>Alternative structure based on signal space</u>

M signals: $s_0(t), \dots, s_{M-1}(t)$

K basis signals: $\phi_0(t), \dots, \phi_{K-1}(t)$

$K \leq M$ (often, $K = 2$)

Then, each of the signal $s_n(t)$ can be written as:

$$s_n(t) = s_{0n}\cdot\phi_0(t) + \dots + s_{(K-1)n}\,\phi_{K-1}(t)$$
$$= \sum_{k=0}^{K-1} s_{kn}\cdot\phi_k(t)$$

$$= \underbrace{\left(s_{0n}, \ldots, s_{(N-1)n}\right)}_{\text{This vector represents the signal}} \cdot \begin{pmatrix} \phi_0(t) \\ \vdots \\ \phi_{N-1}(t) \end{pmatrix}$$

Example:

$$s_1(t) = +\sqrt{\frac{2E}{T}} \cdot \cos(2\pi f_c t) + \sqrt{\frac{2E}{T}} \cdot \sin(2\pi f_c t)$$

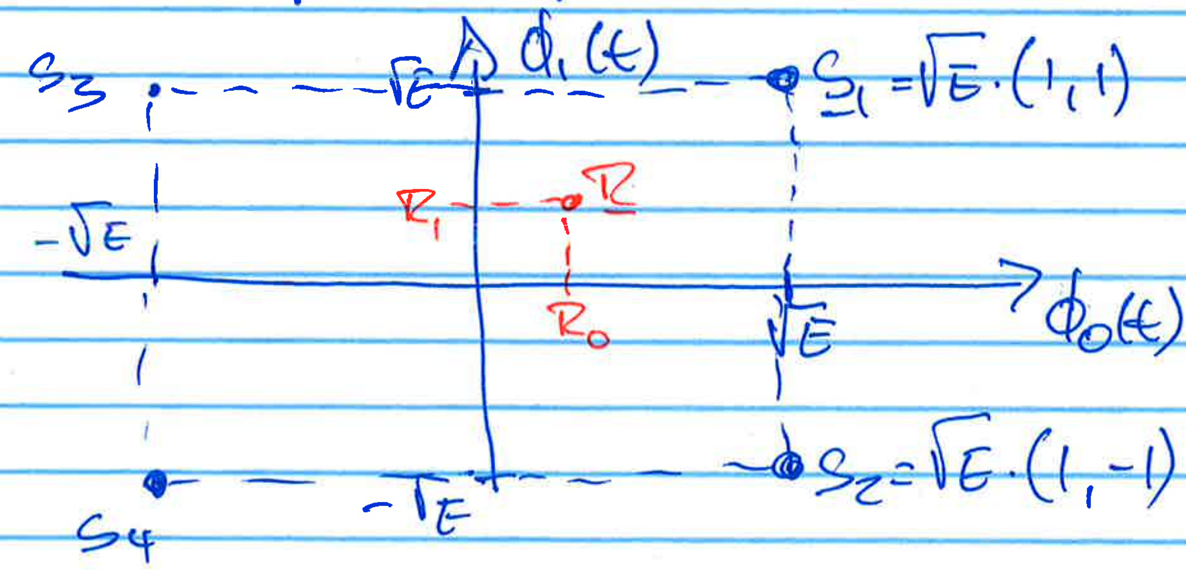
$$s_2(t) = +\sqrt{\frac{2E}{T}} \cdot \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \cdot \sin(2\pi f_c t)$$

$$s_3(t) = -\sqrt{\frac{2E}{T}} \cdot \cos(2\pi f_c t) + \sqrt{\frac{2E}{T}} \cdot \sin(2\pi f_c t)$$

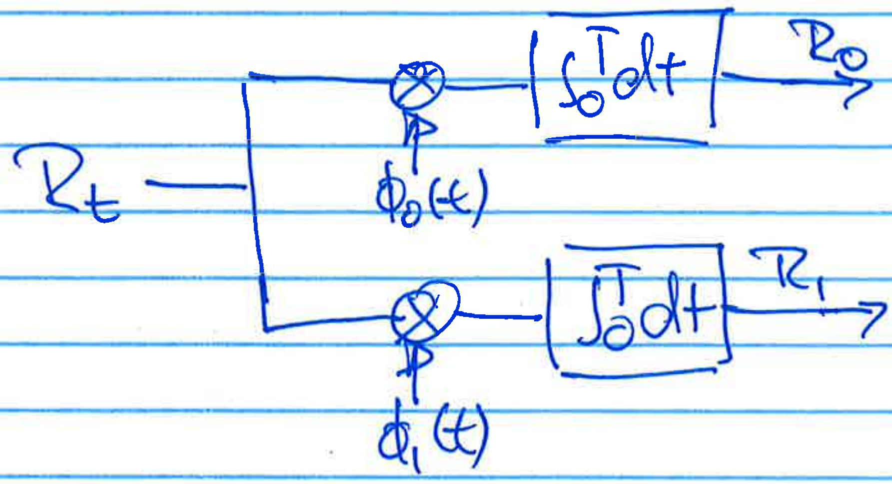
$$s_4(t) = -\sqrt{\frac{2E}{T}} \cdot \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \cdot \sin(2\pi f_c t)$$

$$\phi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad \phi_1(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

QPSK



Receiver frontend:



$$\underline{R} = \begin{pmatrix} R_0 \\ R_1 \end{pmatrix}$$

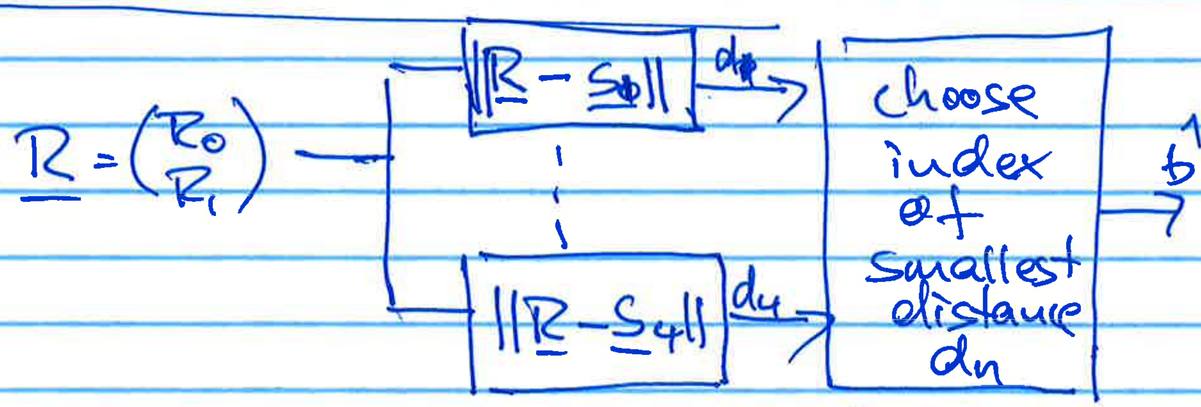
Note: This frontend projects the received signal $R(t)$ into the signal space spanned b $\phi_0(t)$ and $\phi_1(t)$

- conditioned on $s_n(t)$ was transmitted:
  - R_i are Gaussian
  - R_i have variance $\frac{N_0}{2}$
  - $E[R_i | s_n] = \langle s_n(t), \phi_i(t) \rangle$

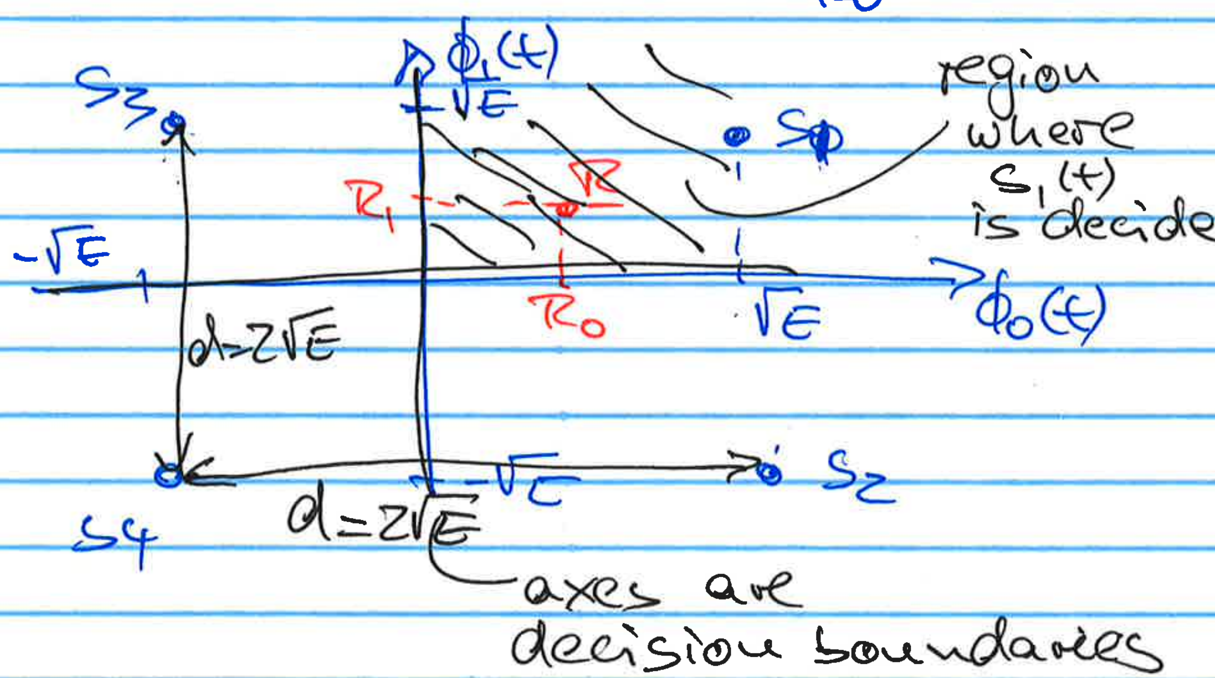
    $= \langle s_n(t), \phi_i(t) \rangle$

    $= s_{in}$
  - R_i, R_j are independent

Optimal Decision Rule:



where $\|\underline{R}-\underline{s}_n\|^2 = \sum_{k=0}^{1} (R_k - s_{nk})^2$



<u>Question:</u> What is the probability of a symbol error

By symmetry:

$$P_{e|s_1} = P_{e|s_2} = P_{e|s_3} = P_{e|s_4} = P_e$$

$\Rightarrow$ arbitrarily pick one, e.g. s_1

$$P_{e|s_1} = \Pr\{\text{decision} \neq s_1 \mid s_1\}$$

With more than one dimension, find probability of correct decision $P_{c|s_1}$ first:

$$P_{c|s_1} = 1 - P_{e|s_1} = \Pr\{\text{decision} = s_1 \mid s_1\}$$

$$= \Pr\{R_0 > 0 \text{ and } R_1 > 0 \mid s_1\}$$

independent R_0, R_1 $\rightarrow$

$$= \Pr\{R_0 > 0 \mid s_1\} \cdot \Pr\{R_1 > 0 \mid s_1\}$$

$$= \left(1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right) \cdot \left(1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

$d = 2\sqrt{E}$

$$= \left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\right)\right)^2$$

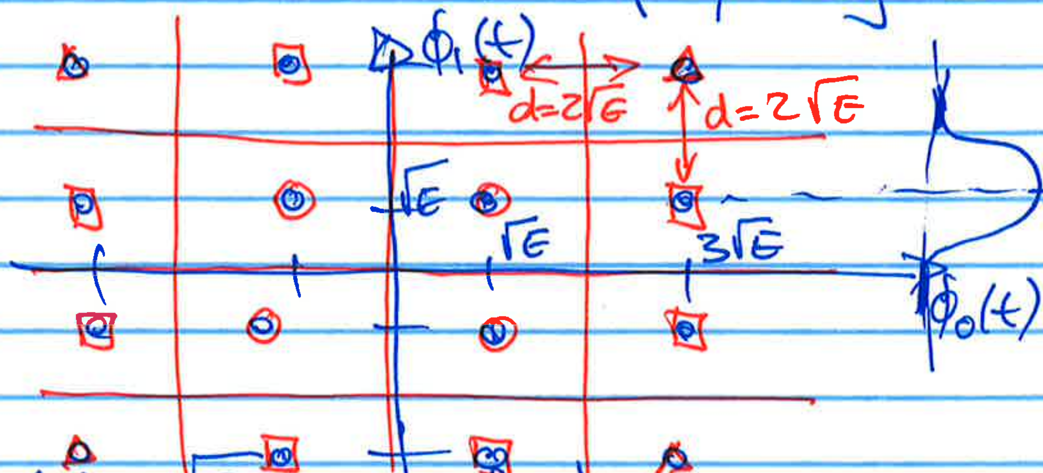
$$\Rightarrow P_e = P_{e|s_1} = 1 - P_{c|s_1} = 1 - \left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\right)\right)^2$$

$$= 2 \cdot Q\left(\sqrt{\frac{2E}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E}{N_0}}\right)$$

Example: 16-QAM

$$s_{n,m} = \sqrt{\frac{2E}{T}} \cdot n \cdot \cos(2\pi f_c t) + \sqrt{\frac{2E}{T}} \cdot m \cdot \sin(2\pi f_c t)$$

with $n, m \in \{\pm 1, \pm 3\}$



$$\phi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cdot \sin(2\pi f_c t)$$

Symmetry: 3 types of signals

△ corners: 4 signal

$$P_{e|\triangle} = 2Q\left(\sqrt{\frac{2E}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E}{N_0}}\right)$$

(same as QPSK)

□ edges: 8 signal

$$P_{c|\square} = \left(1 - 2 \cdot Q\left(\sqrt{\frac{2E}{N_0}}\right)\right) \cdot \left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\right)\right)$$

$$\Rightarrow P_{e|\square} = 3Q\left(\sqrt{\frac{2E}{N_0}}\right) - 2Q^2\left(\sqrt{\frac{2E}{N_0}}\right)$$

o interior: 4 signals

$$P_{c|o} = \left(1-2Q\left(\sqrt{\tfrac{2E}{N_0}}\right)\right)^{2}$$

$$\Rightarrow P_{e|o} = 4Q\left(\sqrt{\tfrac{2E}{N_0}}\right) - 4Q^2\left(\sqrt{\tfrac{2E}{N_0}}\right)$$

<u>Avg. Pe:</u>

$$P_e = \frac{4}{16} P_{e|\triangle} + \frac{8}{16} P_{e|\square} + \frac{4}{16} P_{e|o}$$

$$\vdots$$

$$= 3Q\left(\sqrt{\tfrac{2E}{N_0}}\right) - \frac{9}{4} Q^2\left(\sqrt{\tfrac{2E}{N_0}}\right)$$