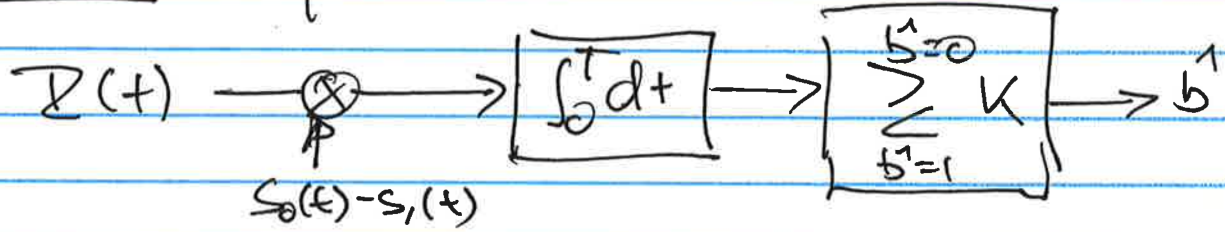
11/28/18

Signal Space Interpretation of Opt. Receiver

Recall: opt. receiver



For $\pi_0=\pi_1=\frac{1}{2}$: + $K=\frac{1}{2}(E_0-E_1)$

+ $P_e=Q\left(\sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}\right)$

+ $P_e=Q\left(\frac{\|s_0(t)-s_1(t)\|}{\sqrt{2N_0}}\right)$

where $\|s_0(t)-s_1(t)\|^2=\int_0^T (s_0(t)-s_1(t))^2\,dt$

Given signal set: $s_0(t)$, $s_1(t)$

Q: Can we find $\phi_0(t)$, $\phi_1(t)$ such that:

1) $\langle \phi_0(t), \phi_1(t)\rangle = 0$

2) $\|\phi_0(t)\| = \|\phi_1(t)\| = 1$

3) $s_0(t) = A_{00}\cdot\phi_0(t) + A_{01}\cdot\phi_1(t)$

$s_1(t) = A_{10}\cdot\phi_0(t) + A_{11}\,\phi_1(t)$

A: Yes, it is always possible to find such $\phi_0(t)$, $\phi_1(t)$

"Proof": The Gram-Schmidt constructs $\phi_0(t), \phi_1(t)$ from $s_0(t), s_1(t)$:

0) $\phi_0(t) = \dfrac{s_0(t)}{\|s_0(t)\|}$ ← for unit energy

1) a) $\psi_1(t) = s_1(t) - \langle s_1(t), \phi_0(t)\rangle \cdot \phi_0(t)$

b) $\phi_1(t) = \dfrac{\psi_1(t)}{\|\psi_1(t)\|}$

Step 1.a) guarantees: $\psi_1(t)$ is orth. to $\phi_0(t)$

$$\langle \psi_1(t), \phi_0(t)\rangle = \langle \underbrace{s_1(t) - \langle s_1(t), \phi_0(t)\rangle \cdot \phi_0(t)}_{\psi_1(t)}, \phi_0(t)\rangle$$

$$= \langle s_1(t), \phi_0(t)\rangle - \langle \underbrace{\langle s_1(t), \phi_0(t)\rangle}_{\text{const.}} \phi_0(t), \phi_0(t)\rangle$$

$$= \langle s_1(t), \phi_0(t)\rangle - \langle s_1(t), \phi_0(t)\rangle \cdot \underbrace{\langle \phi_0(t), \phi_0(t)\rangle}_{\|\phi_0(t)\| = 1}$$

$$= 0$$

To find coefficient A_{ij} in

$$s_0(t) = A_{00} \cdot \phi_0(t) + A_{01}\phi_1(t)$$

$$s_1(t) = A_{10}\phi_0(t) + A_{11} \cdot \phi_1(t)$$

$$A_{00} = \langle s_0(t), \phi_0(t) \rangle$$

$$A_{01} = \langle s_0(t), \phi_1(t) \rangle$$

$$\vdots \qquad\qquad \vdots$$

$$A_{11} = \langle s_1(t), \phi_1(t) \rangle$$

$$\langle s_0(t), \phi_0(t) \rangle = \langle A_{00}\phi_0(t) + A_{01}\phi_1(t), \phi_0(t) \rangle$$

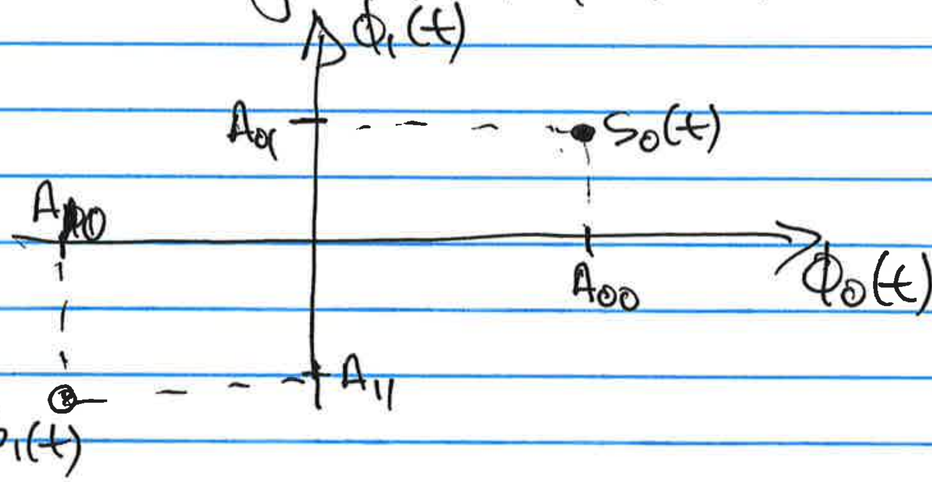
$$= A_{00} \cdot \underbrace{\langle \phi_0(t), \phi_0(t) \rangle}_{=1} + A_{01} \cdot \underbrace{\langle \phi_0(t), \phi_1(t) \rangle}_{=0}$$

$$\Rightarrow \begin{pmatrix} \cancel{s_0(t)} \\ \cancel{s_1(t)} \end{pmatrix}$$

$$s_0(t) = (A_{00}, A_{01}) \cdot \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix}$$

$$s_1(t) = (A_{10}, A_{11}) \cdot \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix}$$

This allows us to interpret signals $s_0(t), s_1(t)$ as "vectors" in a space spanned by $\phi_0(t), \phi_1(t)$



<u>Note:</u> (Parseval's relationship)

$$E_i = \| s_i(t) \|^2 = \int_0^T s_i^2(t)dt = \int_0^T (A_{0i}\phi_0(t) + A_{1i}\phi_1(t))^2 dt$$

$$= \int_0^T A_{0i}^2 \phi_i^2(t) d + 2A_{0i}A_{1i} \underbrace{\cancel{\int \phi_0(t), \phi_1(t) dt}}_{=0}$$

$$+ A_{1i}^2 \int \phi_i^2(t)\, dt$$

$$= A_{0i}^2 + A_{1i}^2 = \| \underline{s}_i \|^2$$

$$\underline{s}_i = \begin{pmatrix} A_{0i} \\ A_{1i} \end{pmatrix}$$

Similarly, $\langle s_0(t), s_1(t) \rangle = \langle \underline{s}_0, \underline{s}_1 \rangle$

Examples:

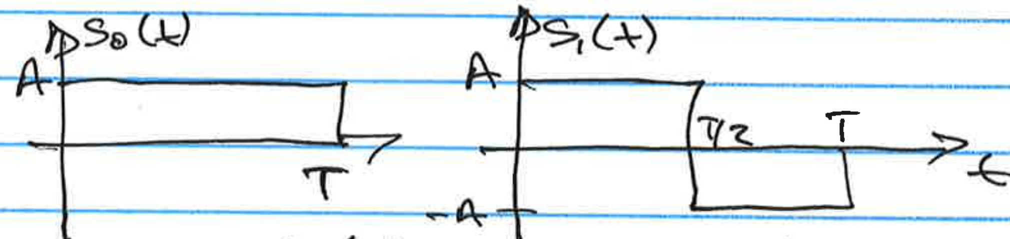
1. $$\underbrace{}_{}s_0(t) = \sqrt{E} \cdot \overbrace{\sqrt{2/T}\,\cos(2\pi f_c t)}^{\phi_0(t)} \qquad 0 \le t \le T$$
$$s_1(t) = \sqrt{E} \cdot \underbrace{\sqrt{2/T}\,\sin(2\pi f_c t)}_{\phi_1(t)}$$

cos(.) and sin(.) are orth.

2.) $$s_0(t) = \sqrt{E} \cdot \overbrace{\sqrt{2/T}\,\cos(2\pi f_0 t)}^{\phi_0(t)}$$
$$s_1(t) = \sqrt{E} \cdot \underbrace{\sqrt{2/T}\,\cos(2\pi f_1 t)}_{\phi_1(t)}$$

$f_0 \cdot T$ and $f_1 \cdot T$ are distinct integers

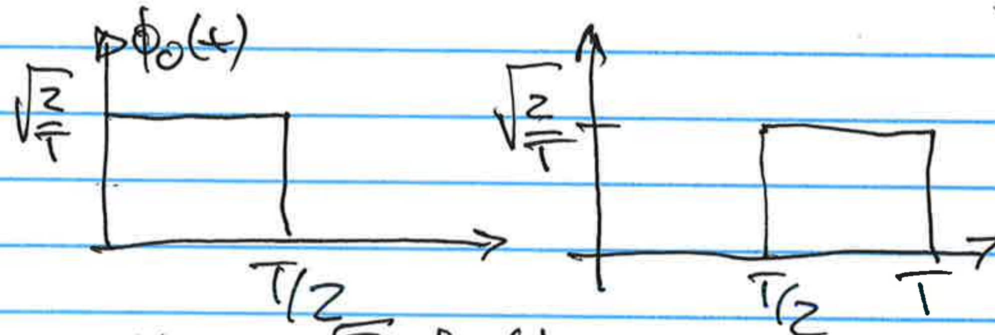
3.)



$$\phi_0(t) = \frac{s_0(t)}{A \cdot \sqrt{T}} \qquad \phi_1(t) = \frac{s_1(t)}{A \cdot \sqrt{T}}$$

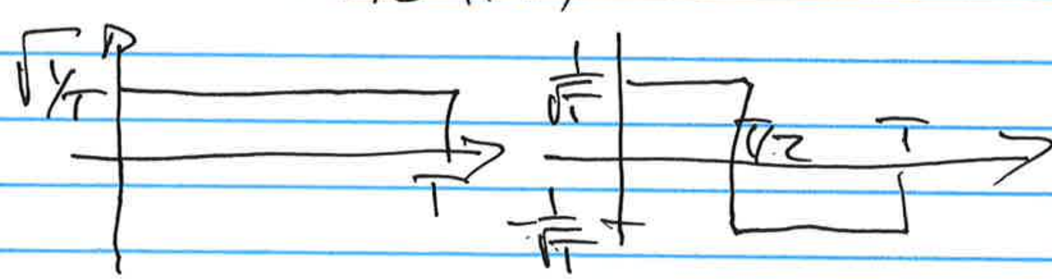
For 3) Possible orthonormal bases:

a)

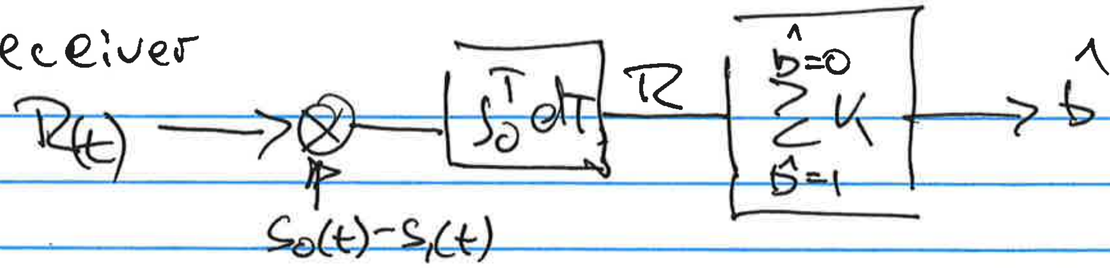


$$s_0(t) = A \cdot \sqrt{T/2}\,\phi_0(t) + A \cdot \sqrt{T/2} \cdot \phi_1(t) \qquad \cancel{s_1(t) =}$$

b)



opt. receiver



$$R = \langle R(t),\ s_0(t)-s_1(t)\rangle$$

$$= \langle R(t),\ s_0(t)\rangle - \langle R(t),\ s_1(t)\rangle$$

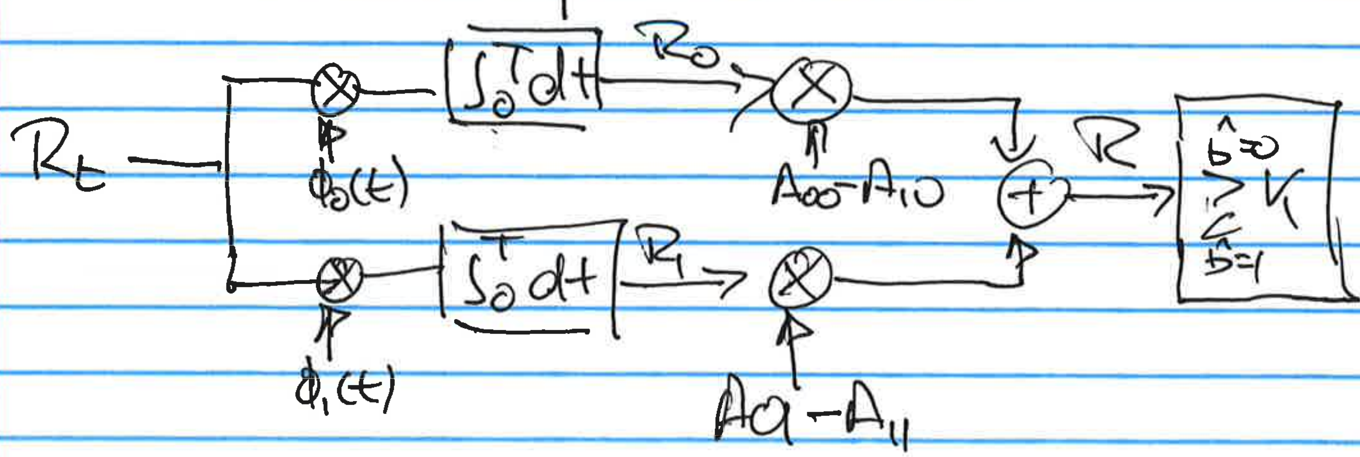
$$= \langle R(t),\ A_{00}\phi_0(t)+A_{01}\cdot\phi_1(t)\rangle - \langle R(t),\ A_{10}\phi_0(t)+A_{11}\phi_1(t)\rangle$$

$$= A_{00}\cdot\langle R(t),\underline{\phi_0}(t)\rangle + A_{01}\langle R(t),\phi_1(t)\rangle - A_{10}\langle R(t),\phi_0(t)\rangle - A_{11}\langle R(t),\phi_1(t)\rangle$$

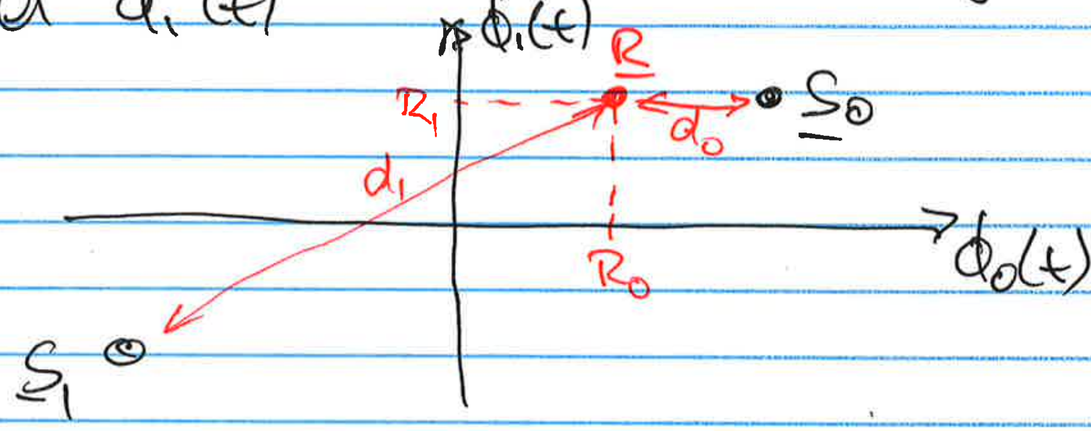
$$= (A_{00}-A_{10})\cdot\underbrace{\langle R(t),\ \phi_0(t)\rangle}_{R_0} + (A_{01}-A_{11})\underbrace{\langle R(t),\phi_1(t)\rangle}_{R}$$

$$= \begin{pmatrix} A_{00}-A_{10} \\ A_{01}-A_{11}\end{pmatrix}^T \cdot \underbrace{\begin{pmatrix} R_0 \\ R_1\end{pmatrix}}_{\underline{R}} = \langle \underline{s}_0-\underline{s}_1,\ \underline{R}\rangle$$

$\Rightarrow$ alternative opt. receiver



Frontend projects the received signal into signal space spanned by $\phi_0(t)$ and $\phi_1(t)$



For $\pi_0 = \pi_1 = \frac{1}{2}$, opt. receiver decides in favor of signal that is closest to $\underline{R}$:

$$d_0 \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} d_1$$

$$\Leftrightarrow \quad \|\underline{R} - \underline{s}_0\|^2 \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} \|\underline{R} - \underline{s}_1\|^2$$

$$\Leftrightarrow \quad \cancel{\|\underline{R}\|^2} - 2\langle \underline{R}, \underline{s}_0 \rangle + \|\underline{s}_0\|^2 \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} \cancel{\|\underline{R}\|^2} - 2\langle \underline{R}, \underline{s}_1 \rangle + \|\underline{s}_1\|^2$$

$$\Leftrightarrow \quad \|\underline{s}_0\|^2 - \|\underline{s}_1\|^2 \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} 2\langle \underline{R}, \underline{s}_0 \rangle - 2\langle \underline{R}, \underline{s}_1 \rangle$$

$$\frac{\|\underline{s}_0\|^2 - \|\underline{s}_1\|^2}{2} \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} \langle \underline{R}, \underline{s}_0 - \underline{s}_1 \rangle$$

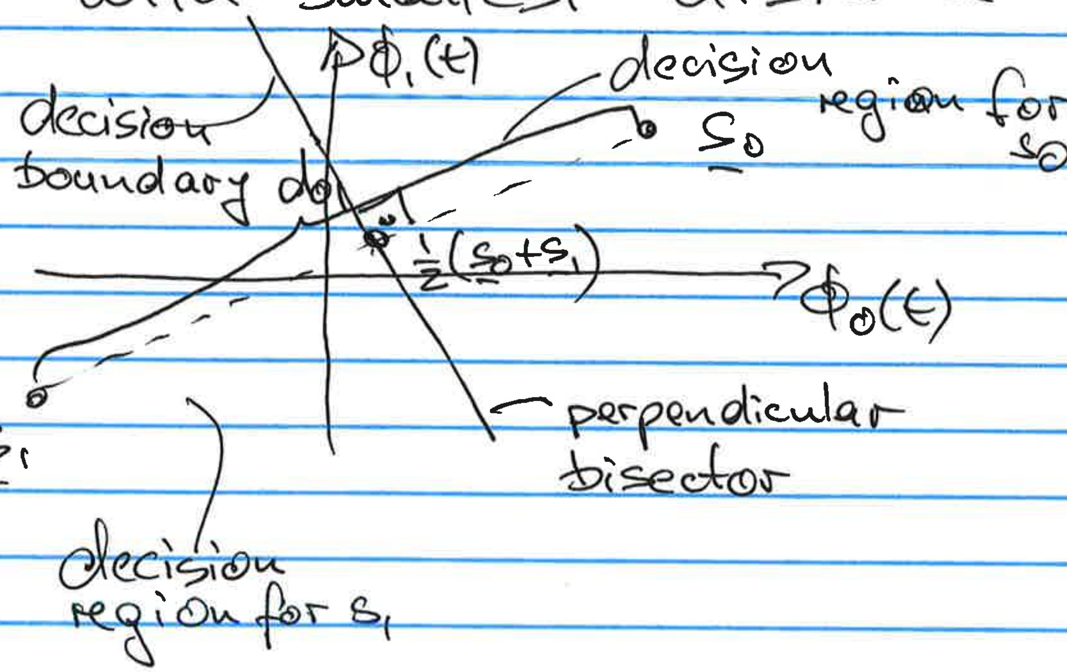
OPTIMUM RECEIVER!

$$K = \frac{\|\underline{s}_0\|^2 - \|\underline{s}_1\|^2}{2} \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} \langle R_t, s_0(t) - s_1(t) \rangle$$

Interpretation: of opt receiver

1. Frontend: projects the received signal into the signal space that contains $s_0(t)$, $s_1(t)$ (spanned by $\phi_0(t)$, $\phi_1(t)$)

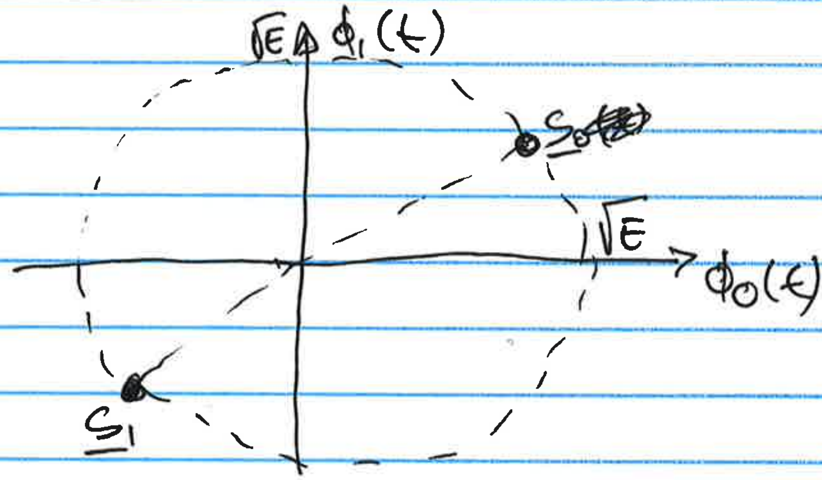
2. Decision: is based on distances $d_i = \|R - s_i\|$ between received signal and signals $s_0(t)$, $s_1(t)$. Decision is in favor of signal with smallest distance



Finally, $P_e = Q\left(\frac{\|s_0 - s_1\|}{\sqrt{2N_0}}\right)$

$$= Q\left(\frac{d_{01}}{\sqrt{2N_0}}\right)$$

Q: If signals are limited to energy E, what is the best (min P_e) signal set $s_0(t)$, $s_1(t)$?



A: Signals have to be negatives of each other and lie on circle with radius $\sqrt{E}$

$\Rightarrow$ this is the BPSK signal set.