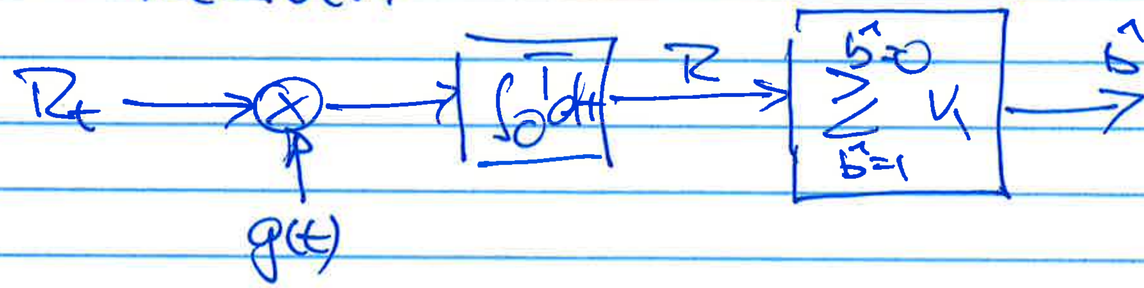
11/26/18

# Linear Receivers

Setup as before:

- binary signal set: $s_0(t)$, $s_1(t)$
- a priori prob.: $\pi_0, \pi_1$
- AWGN spectral height $\frac{N_0}{2}$

Want: Find $P_e$ for general linear receiver:




Plan:

1.) Find conditional pdfs of $R$: $f_{R|b=0}(r)$, $f_{R|b=1}(r)$

2.) Conditional $P_e$: $\Pr\{\hat{b}=1 \mid b=0\}$
$\Pr\{\hat{b}=0 \mid b=1\}$

3.) Avg. $P_e$: $\pi_0 \cdot \Pr\{\hat{b}=1 \mid b=0\} + \pi_1 \cdot \Pr\{\hat{b}=0 \mid b=1\}$

1. <u>Conditional pdfs:</u>

a) assume $b=0 \Rightarrow s_0(t)$ was sent

- $R$ is Gaussian

- $E[R|b=0] = \int_0^T s_0(t) \cdot g(t)\, dt$

$= \langle s_0(t), g(t) \rangle$ "Inner Product"

- $Var[R|b=0] = Var\left[\int_0^T N_t \cdot g(t)\, dt\right]$

$= \frac{N_0}{2} \int_0^T g^2(t)\, dt$

$= \frac{N_0}{2} \|g(t)\|^2$ $\left(\frac{\text{Norm}}{\text{squared}}\right)$

b) Similarly, for $b=1$

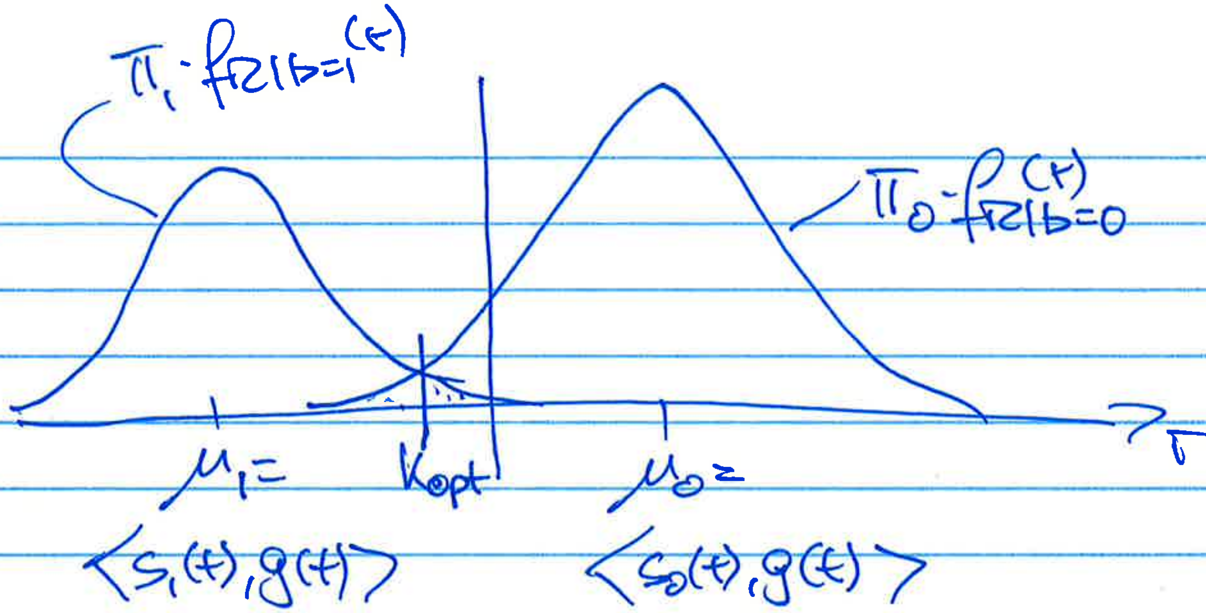
- $E[R|b=1] = \langle s_1(t), g(t) \rangle$

- $Var[R|b=1] = \frac{N_0}{2} \|g(t)\|^2$

Summary:

$f_{R|b=0}^{(r)} \sim N\left(\langle s_0(t), g(t) \rangle, \frac{N_0}{2} \|g(t)\|^2\right)$

$f_{R|b=1}^{(r)} \sim N\left(\langle s_1(t), g(t) \rangle, \frac{N_0}{2} \|g(t)\|^2\right)$




$K_{opt}$ is the solution of

$$\pi_0 \cdot f_{R|b=0}(K_{opt}) = \pi_1 \cdot f_{R|b=1}(K_{opt})$$

For the special case: $\pi_0 = \pi_1 = \frac{1}{2}$:

$$\boxed{\begin{aligned} K_{opt} &= \frac{1}{2}\left(\langle s_0(t), g(t) \rangle + \langle s_1(t), g(t) \rangle\right) \\ &= \left\langle \frac{s_0(t) + s_1(t)}{2}, g(t) \right\rangle \end{aligned}}$$

## 2. Conditional error probability

Assume $b=1$:

$$\Pr\{\hat{b}=0 \mid b=1\} = \Pr\{R>0 \mid b=1\}$$

$$= \int_K^{\infty} f_{R|b=1}(r)\, dr$$

$$= \int_K^{\infty} \frac{1}{\sqrt{2\pi \cdot \underbrace{\frac{N_0}{2}\|g(t)\|^2}_{\sigma^2}}} \cdot \exp\left(-\frac{1}{2} \frac{(r-\overbrace{\langle s_1(t), g(t)\rangle}^{\mu_1})^2}{\underbrace{\frac{N_0}{2}\|g(t)\|^2}_{\sigma^2}}\right) dr$$

$$z = \frac{r-\langle s_1(t), g(t)\rangle}{\sqrt{\frac{N_0}{2}\|g(t)\|^2}}: \quad = \int_{\frac{K-\langle s_1(t), g(t)\rangle}{\sqrt{\frac{N_0}{2}\|g(t)\|^2}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= Q\left(\frac{K-\langle s_1(t), g(t)\rangle}{\sqrt{\frac{N_0}{2} \cdot \|g(t)\|^2}}\right)$$

Recall (for $\pi_0=\pi_1=\frac{1}{2}$) ~~$K_{opt} =$~~

$$K_{opt} = \frac{1}{2}\left(\langle s_0(t), g(t)\rangle + \langle s_1(t), g(t)\rangle\right)$$

$$\Rightarrow P_e = Q\left(\frac{\frac{1}{2}\left(\langle s_0(t), g(t)\rangle - \langle s_1(t), g(t)\rangle\right)}{\sqrt{\frac{N_0}{2} \cdot \|g(t)\|^2}}\right)$$

$$= Q\left(\frac{\langle s_0(t)-s_1(t), g(t)\rangle}{\sqrt{2N_0 \cdot \|g(t)\|^2}}\right)$$

Similarly for ~~the~~ $b=0$:

$$\Pr\{\hat{b}=1 \mid b=0\} = Q\left(\frac{\langle s_0(t), g(t)\rangle - K}{\sqrt{\frac{N_0}{2}\,\|g(t)\|^2}}\right)$$

For $\pi_0=\pi_1=\frac{1}{2}$ and $K_{opt}=\frac{1}{2}\langle s_0(t)+s_1(t), g(t)\rangle$

$$\Rightarrow \Pr\{\hat{b}=1 \mid b=0\} = Q\left(\frac{\langle s_0(t)-s_1(t), g(t)\rangle}{\sqrt{2N_0\cdot\|g(t)\|^2}}\right)$$

$$= \Pr\{\hat{b}=0 \mid b=1\}$$

3.) Average $P_e$:

For equal likely signals $(\pi_0=\pi_1=\frac{1}{2})$

$$\boxed{P_e = Q\left(\frac{\langle s_0(t)-s_1(t), g(t)\rangle}{\sqrt{2N_0\cdot\|g(t)\|^2}}\right)}$$

Question: Which choice of $g(t)$ minimizes the probability of error?

$$P_e = Q\left( \frac{\langle s_0(t) - s_1(t), g(t) \rangle}{\sqrt{2N_0 \cdot \|g(t)\|^2}} \right)$$

Equivalent question: which $g(t)$ maximizes:

$$\frac{\langle s_0(t) - s_1(t), g(t) \rangle}{\sqrt{2N_0 \cdot \|g(t)\|^2}}$$

Schwartz inequality for inner products

$$\langle a, b \rangle \leq \|a\| \cdot \|b\|$$

with equality if and only if

$$a = \underbrace{c}_{c>0,\ \text{scalar}} \cdot b$$

$\Rightarrow P_e$ is minimized if

$$g(t) = c \cdot (s_0(t) - s_1(t))$$

$$\Rightarrow P_e = Q\left( \frac{c \cdot \|s_0(t) - s_1(t)\|^2}{\sqrt{2N_0 \cdot \|s_0(t) - s_1(t)\|^2 \cdot c^2}} \right)$$

$$\boxed{P_e = Q\left( \sqrt{\frac{\|s_0(t) - s_1(t)\|^2}{2N_0}} \right)}$$

Summary:

1.) optimum choice for $g(t)$:

$$g(t) = c_0 \left( s_0(t) - s_1(t) \right)$$

2.) Resulting $P_e$ (for $\pi_0 = \pi_1$)

$$P_e = Q\left( \frac{\| s_0(t) - s_1(t) \|}{\sqrt{2 N_0}} \right)$$

Note: $\| s_0(t) - s_1(t) \|^2$

$$= \langle s_0(t) - s_1(t), s_0(t) - s_1(t) \rangle$$

$$= \langle s_0, s_0 \rangle - 2 \langle s_0, s_1 \rangle + \langle s_1, s_1 \rangle$$

$$= \| s_0 \|^2 - 2 \langle s_0, s_1 \rangle + \| s_1 \|^2$$

$$= \int_0^T s_0^2(t)\,dt - 2 \int_0^T s_0(t) s_1(t)\,dt + \int_0^T s_1^2(t)\,dt$$

$$= E_0 - 2 \cdot r_{01} + E_1$$

$$\Rightarrow P_e = Q\left( \sqrt{\frac{E_0 + E_1 - 2 r_{01}}{2 N_0}} \right)$$

Q: ~~Whi~~ For fixed $E_0, E_1$, which choice of $r_{01}$ ($< \sqrt{E_0 \cdot E_1}$) minimizes $P_e$?

$$P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2r_{01}}{2N_0}}\right)$$

For $r_{01} = 0$: $P_e = Q\left(\sqrt{\frac{E}{N_0}}\right)$

$E_0 = E_1 = E$

Bad

same signal

For $r_{01} = E$: $P_e = Q(0) = \frac{1}{2}$

$E_0 = E_1 = r_{01} = E$

For $r_{01} = -E$ $\boxed{P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)}$

$E_0 = E_1 = E$