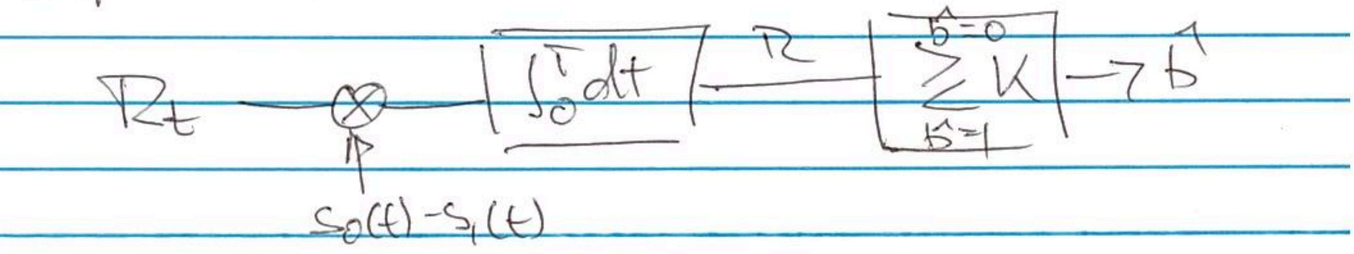
11/19

From last time:

opt. receivers:



Goal: Find average Prob. of error P_e

1.) Find conditional pdfs of R (given $b=0$, $b=1$)

Given $b=0$:

$$R \sim N(\underbrace{E_0 - r_{01}}_{\mu_0}, \underbrace{\tfrac{N_0}{2}\cdot(E_0+E_1-2r_{01})}_{\sigma^2})$$

$b=1$: $R \sim N(\underbrace{r_{01}-E_1}_{\mu_1}, \frac{N_0}{2}(E_0+E_1-2r_{01}))$

with $E_0 = \int_0^T s_0^2(t)\,dt \quad r_{01} = \int_0^T s_0(t)\cdot s_1(t)\,dt$

$E_1 = \int_0^T s_1^2(t)\,dt$

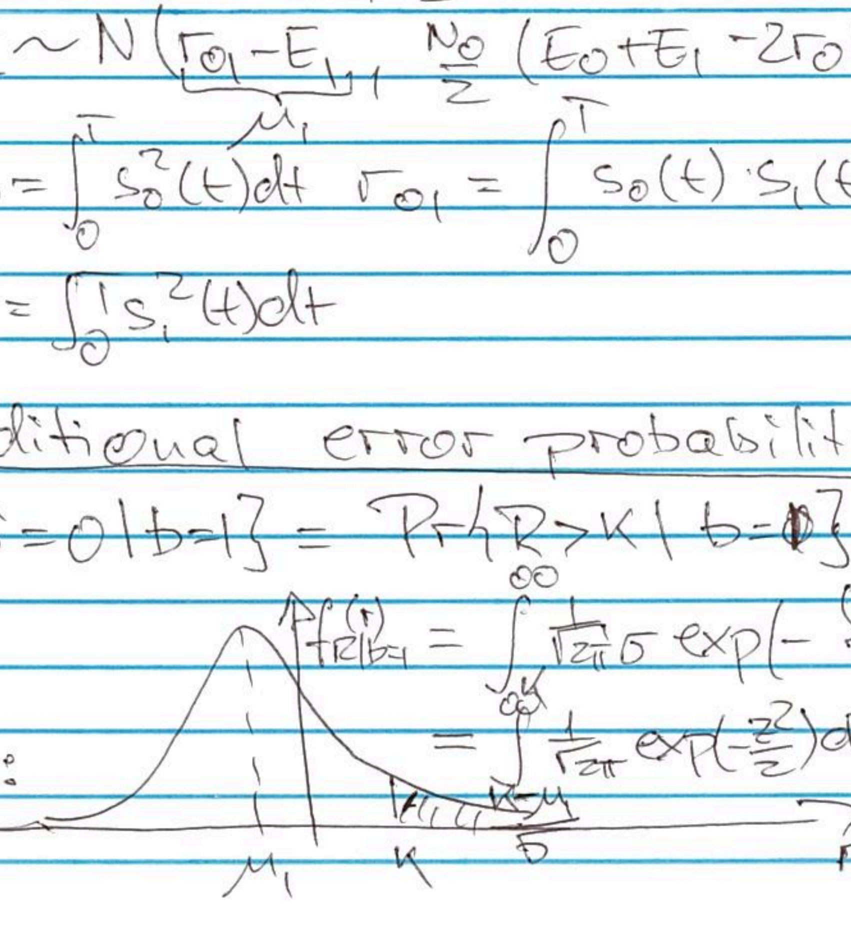
2. Conditional error probabilities

$$\Pr\{\hat{b}=0 \mid b=1\} = \Pr\{R>K \mid b=1\}$$

$$= \int_K^\infty \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(r-\mu_1)^2}{2\sigma^2}\right) dr$$

Subst: $z = \frac{r-\mu_1}{\sigma}$:

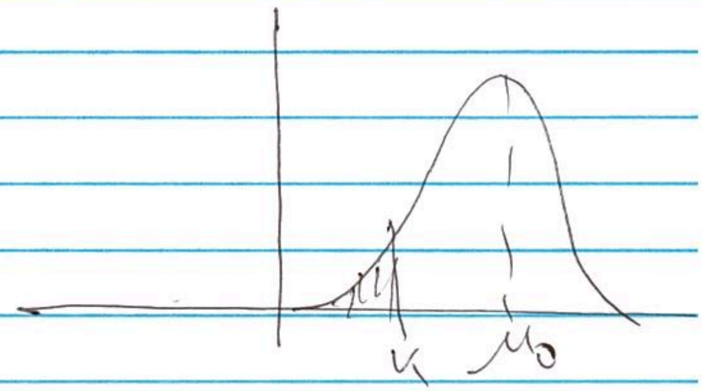
$$= \int_{\frac{K-\mu_1}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = Q\left(\frac{K-\mu_1}{\sigma}\right)$$



Similarly, $\Pr\{\hat{b}=1 \mid b=0\}$

$$= \Pr\{R < K \mid b=0\}$$

$$= Q\left(\frac{\mu_0 - K}{\sigma}\right)$$

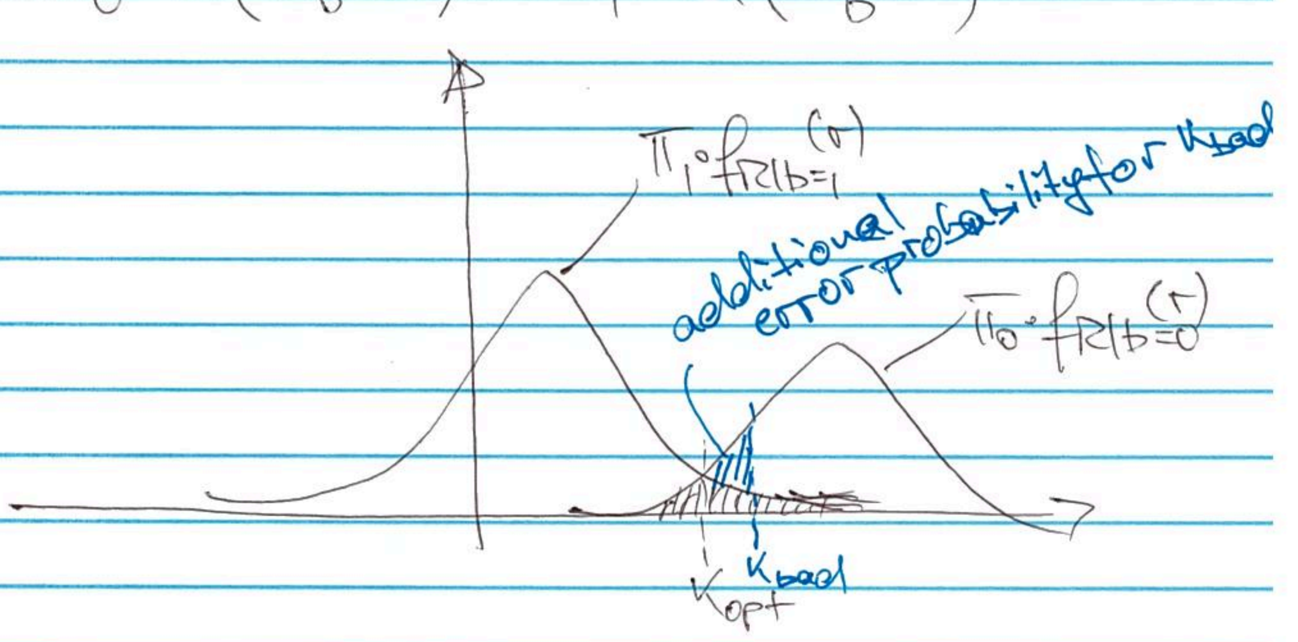


3.) Avg. Pe

$$P_e = \Pr\{\hat{b} = b\}$$

$$= \Pr\{\hat{b}=1 \mid b=0\} \cdot \pi_0 + \Pr\{\hat{b}=0 \mid b=1\} \cdot \pi_1$$

$$= \pi_0 \cdot Q\left(\frac{\mu_0 - K}{\sigma}\right) + \pi_1 \cdot Q\left(\frac{K - \mu_1}{\sigma}\right)$$



$\Rightarrow$ For optimal threshold K_{opt}:

$$\pi_0 \cdot f_{R|b=0}(K_{opt}) = \pi_1 \cdot f_{R|b=1}(K_{opt})$$

$$\Rightarrow \frac{\pi_0}{\pi_1} = \frac{f_{R|b=1}(r)}{f_{R|b=0}(r)} \quad \text{likelihood ratio}$$

$$\Rightarrow \pi_0 \cdot \cancel{\frac{1}{\sqrt{2\pi}\sigma}} \exp\left(-\frac{(K_{opt}-\mu_0)^2}{2\sigma^2}\right) = \pi_1 \cdot \cancel{\frac{1}{\sqrt{2\pi}\sigma}} \exp\left(-\frac{(K_{opt}-\mu_1)^2}{2\sigma^2}\right)$$

Take $\ln(\cdot)$: $$\ln \pi_0 + \underbrace{\left(-\frac{(K_{opt}-\mu_0)^2}{2\sigma^2}\right)}_{\frac{\cancel{K_{opt}^2} - 2\mu_0 \cdot K_{opt} + \mu_0^2}{2\cdot\sigma^2}} = \ln \pi_1 + \underbrace{\left(-\frac{(K_{opt}-\mu_1)^2}{2\sigma^2}\right)}_{\frac{\cancel{K_{opt}^2} - 2K_{opt}\mu_1 + \mu_1^2}{2\sigma^2}}$$

$$2\sigma^2(\ln \pi_0 - \ln \pi_1) = 2K_{opt} \cdot (\mu_0 - \mu_1) - (\mu_0^2 - \mu_1^2)$$

$$\Rightarrow 2\sigma^2 \ln \frac{\pi_0}{\pi} + (\mu_0^2 - \mu_1^2) = 2K_{opt}(\mu_0 - \mu_1)$$

$$\Rightarrow \frac{\sigma^2}{\mu_0 - \mu_1} \cdot \ln \frac{\pi_0}{\pi_1} + \frac{\mu_0^2 - \mu_1^2}{2(\mu_0 - \mu_1)} = K_{opt}$$

$$\Rightarrow \boxed{\frac{\sigma^2}{\mu_0 - \mu_1} \ln \frac{\pi_0}{\pi_1} + \frac{\mu_0 + \mu_1}{2} = K_{opt}}$$

Note: when $\pi_0 = \pi = \frac{1}{2} \Rightarrow K_{opt} = \frac{\mu_0 + \mu_1}{2}$

$$K_{opt} = \frac{2\sigma^2 \cdot \ln\frac{\pi_1}{\pi_0} + (\mu_0^2 - \mu_1^2)}{2(\mu_0 - \mu_1)}$$

$$= \frac{\sigma^2}{\mu_0 - \mu_1} \cdot \ln\frac{\pi_1}{\pi_0} + \frac{1}{2}\frac{\mu_0^2 - \mu_1^2}{\mu_0 - \mu_1}$$

$$= \frac{\sigma^2}{\mu_0 - \mu_1} \cdot \ln\frac{\pi_1}{\pi_0} + \frac{1}{2}(\mu_0 + \mu_1)$$

With:

$\mu_0 = E_0 - r_{01}$ $\quad \mu_1 = r_{01} - E_1$ $\quad \sigma^2 = \frac{N_0}{2} \cdot (E_0 + E_1 - 2r_{01})$

$$K_{opt} = \frac{N_0}{2} \cdot \frac{E_0 + E_1 - 2r_{01}}{E_0 - r_{01} - (r_{01} - E_1)} \ln\frac{\pi_1}{\pi_0} + \frac{1}{2}(E_0 - r_{01} + r_{01} - E_1)$$

$$\boxed{K_{opt} = \frac{N_0}{2} \cdot \ln\frac{\pi_1}{\pi_0} + \frac{1}{2} \cdot (E_0 - E_1)}$$

Important special case:

$$\pi_0 = \pi_1 = \frac{1}{2} : \quad K_{opt} = \frac{1}{2}(E_0 - E_1)$$

<u>For this special case:</u>

$$\frac{\mu_0 - K}{\sigma} = \frac{(E_0 - r_{01}) - \frac{1}{2}(E_0 - E_1)}{\sqrt{\frac{N_0}{2} \cdot (E_0 + E_1 - 2r_{01})}}$$

$$= \frac{1}{2} \cdot \frac{(E_0 + E_1 - 2r_{01})}{\sqrt{\frac{N_0}{2}(E_0 + E_1 - 2r_{01})}} = \sqrt{\frac{E_0 + E_1 - 2r_{01}}{2N_0}}$$

with $\mu_0 = E_0 - r_{01} \qquad \mu_1 = \cancel{E_1} \cdot r_{01} - E_1$

$$\sigma^2 = \frac{N_0}{2}\left(E_0 + E_1 - 2r_{01}\right)$$

$$K_{opt} = \frac{\frac{N_0}{2}\left(E_0 + E_1 - 2r_{01}\right)}{\left(E_0 - r_{01}\right) - \left(r_{01} - E_1\right)} \cdot \ln\frac{\pi_0}{\pi_1} + \frac{\left(E_0 - r_{01}\right) + \left(r_{01} - E_1\right)}{2}$$

$$\boxed{K_{opt} = \frac{N_0}{2}\ln\frac{\pi_0}{\pi_1} + \frac{E_0 - E_1}{2}}$$

Important special case: $\pi_0 = \pi_1 = \frac{1}{2}$

$$\Rightarrow K_{opt} = \frac{E_0 - E_1}{2}$$

<u>For this special case:</u>

$$\frac{\mu_0 - K}{\sigma} = \frac{K - \mu_1}{\sigma} = \frac{\left(E_0 - r_{01}\right) - \left(\frac{E_0 - E_1}{2}\right)}{\sqrt{\frac{N_0}{2}\left(E_0 + E_1 - 2r_{01}\right)}}$$

$$= \frac{\frac{1}{2}\left(E_0 + E_1 - 2r_{01}\right)}{\sqrt{\frac{N_0}{2}\left(E_0 + E_1 - 2r_{01}\right)}}$$

$$= \sqrt{\frac{\left(E_0 + E_1 - 2r_{01}\right)}{2N_0}}$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2r_{01}}{2N_0}}\right)$$

$$\frac{K-\mu_1}{\sigma} = \frac{\frac{1}{2}(E_0-E_1) - (r_{01}-E_1)}{\sqrt{\frac{N_0}{2}(E_0+E_1-2r_{01})}}$$

$$= \frac{\frac{1}{2}(E_0+E_1-2r_{01})}{\sqrt{\frac{N_0}{2}(E_0+E_1-2r_{01})}} = \sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}$$

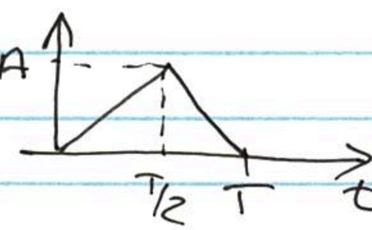
$$\Rightarrow P_e = \pi_0 \cdot Q\left(\sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}\right) + \pi_1 \cdot Q\left(\sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}\right)$$

$$\boxed{P_e = Q\left(\sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}\right)}$$

Examples:

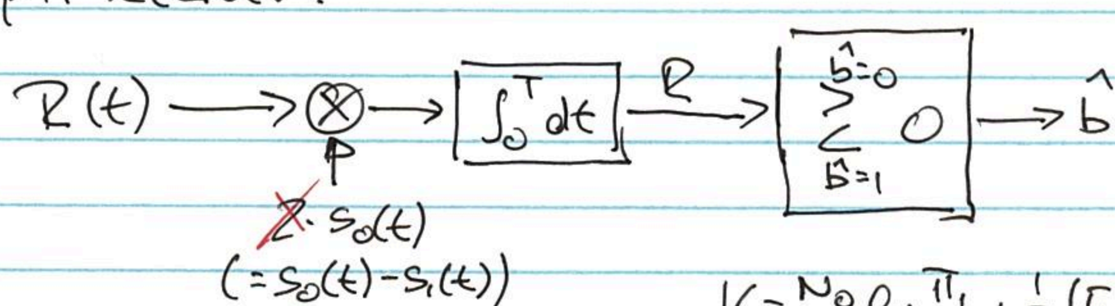
BPSK: $s_0(t) = \begin{cases} 2A\, t/T & 0 \le t \le \frac{T}{2} \\ 2A - 2At/T & T/2 \le t \le T \end{cases}$

$s_1(t) = -s_0(t)$



Let $\pi_0 = \pi_1 = \frac{1}{2}$

Opt. receiver:

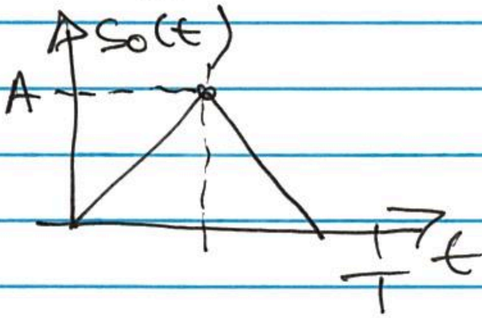


$$K = \frac{N_0}{2}\ln\frac{\pi_1}{\pi_0} + \frac{1}{2}(E_0-E_1)$$
$$= 0$$

Example 1: BPSK w/ triangular pulse

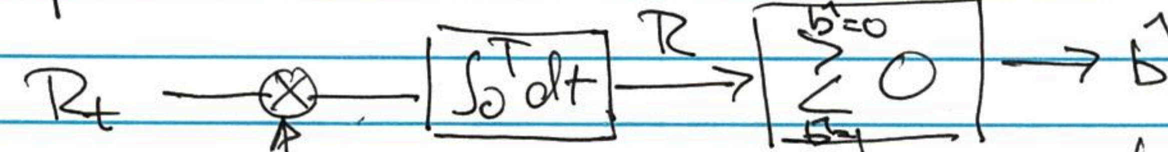
$$s_0(t) = \begin{cases} 2At/T & 0 \le t < T/2 \\ 2A - 2t/T & T/2 \le t < T \\ 0 & \text{else} \end{cases}$$

$$s_1(t) = -s_0(t)$$



$$\pi_0 = \pi_1 = \frac{1}{2}$$

opt. receiver:



$s_0(t) - s_1(t) = 2 \cdot s_0(t)$

$$K_{opt} = \underbrace{\cancel{\frac{N_0}{2} \ln \frac{\pi_0}{\pi_1}}}_{=0 \text{ for } \pi_0 = \pi_1} + \underbrace{\cancel{\frac{1}{2}(E_0 - E_1)}}_{=0}$$

signals are equal energy

$$\Rightarrow K_{opt} = 0$$

$$P_e = Q\left(\sqrt{\frac{(E_0 + E_1 - 2r_{01})}{2N_0}}\right)$$

Need: $E_0 = \int_0^T s_0^2(t)\,dt = 2 \cdot \int_0^{T/2} (2At/T)^2\,dt$

$$= 2 \cdot \left(\frac{2A}{T}\right)^2 \cdot \int_0^{T/2} t^2\,dt$$

$$= \frac{8A^2}{T^2} \cdot \frac{t^3}{3}\Big|_0^{T/2}$$

$$= \frac{8A^2}{T^2} \cdot \frac{T^3}{3 \cdot 8} = \frac{A^2 \cdot T}{3}$$

Siniliarly, $E_1 = \int_0^T s_1^2(t)\,dt$

$$= \int_0^T (-s_0(t))^2\,dt$$

$$= \int_0^T s_0^2(t)\,dt = E_0 = \frac{A^2T}{3}$$

$$r_{01} = \int_0^T s_0(t)\cdot s_1(t)\,dt$$

$$= \int_0^T s_0(t)\cdot(-s_0(t))\,dt = -E_0$$

$$= -\frac{A^2T}{3}$$

$$P_e = Q\left(\sqrt{\frac{(E_0+E_1-2r_{01})}{2N_0}}\right) = Q\left(\sqrt{\frac{4\cdot E_0}{2N_0}}\right)$$

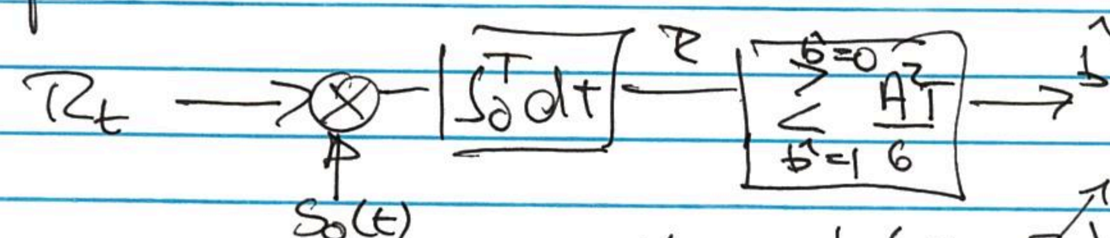
$$= Q\left(\sqrt{\frac{2E_0}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{3N_0}}\right)$$

Example 2: On-Off Keying (OOK)

$$s_0(t) = \begin{cases} 2At/T & 0 \le t \le T/2 \\ 2A - 2At/T & T/2 \le t \le T \\ 0 & \text{else} \end{cases}$$

$$s_1(t) = 0$$

opt. receiver:



$$K_{opt} = \frac{1}{2}(E_0 - \cancel{E_1}^{\,=0})$$

$$= \frac{1}{2}E_0 = \frac{A^2 \cdot T}{6}$$

$$P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2r_{01}}{2N_0}}\right)$$

$$E_0 = \frac{A^2T}{3}$$

$$E_1 = 0$$

$$r_{01} = \int_0^T s_0(t)\, s_1(t)\, dt = 0$$

$$= Q\left(\sqrt{\frac{A^2T}{6N_0}}\right)$$