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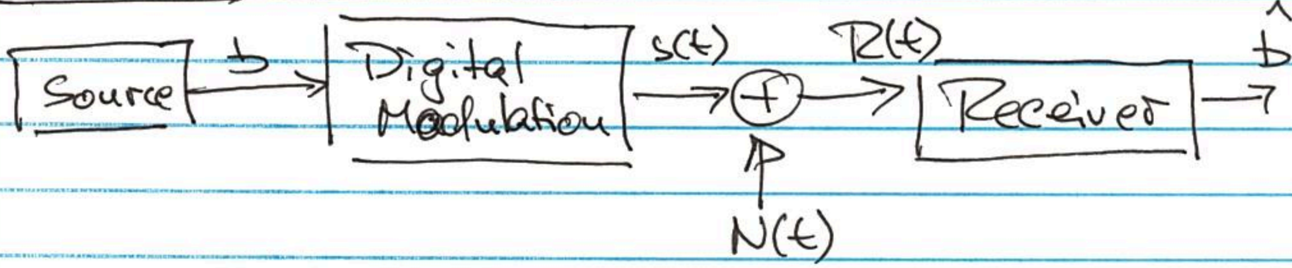
# Optimum Receivers for Binary Signals

**Context:**




**Source:** $b=0$ with prob. $\pi_0$
$b=1$ with prob $\pi_1$

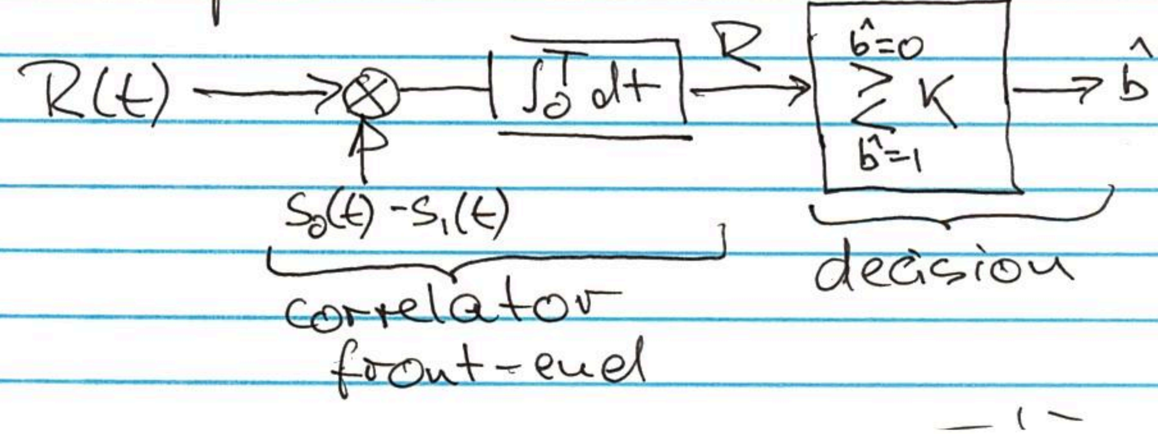
**Modulator:** If $b=0$, then $s(t) = s_0(t)$
If $b=1$, then $s(t) = s_1(t)$

**Note:** duration of $s_0(t)$, $s_1(t)$ is $T$.

**Channel:** AWGN, spectral height $\frac{N_0}{2}$

$$\Rightarrow \text{ACF}: R_N(\tau) = \frac{N_0}{2}\delta(\tau)$$

**Receiver:** It can be shown that the optimum receiver is



with optimum threshold K given by

$$K = \frac{N_0}{2} \ln \frac{\pi_1}{\pi_0} + \frac{1}{2}(E_0 - E_1)$$

where $E_0 = \int_0^T s_0^2(t)\,dt \qquad E_1 = \int_0^T s_1^2(t)\,dt$

<u>Goal:</u> Find a general expression for the probability of error for optimum receiver.

<u>Plan:</u>

① Find conditional pdfs of R given $s_0(t)$ and $s_1(t)$, respectively. ✓

② Find conditional probabilité of error:

$$P_r(R > K \mid b = 1)$$

$$P_r\{R < K \mid b = 0)$$

③ average Pe

$$P_e = \pi_0 \cdot P_r\{R < K \mid b = 0\} + \pi_1 \cdot P_r\{R > K \mid b = 1\}$$

## ① conditional pdfs of R

a) assume $b=0'$ $\Rightarrow$ $s_0'(t)$ was sent

$$\Rightarrow R(t) = s_0'(t) + N(t)$$

$$\Rightarrow R = \int_0^T R(t)\cdot\left(s_0(t)-s_1(t)\right)dt$$

$$= \int_0^T \left(s_0'(t)+N_t\right)\cdot\left(s_0(t)-s_1(t)\right)dt$$

$\Rightarrow R$ is Gaussian (conditional on $s_0'(t)$)

$\rightarrow$ Find mean and variance

$$E\left[R \mid s_0'(t)\right] = E\left[\int_0^T \left(s_0'(t)+N(t)\right)\cdot\left(s_0(t)-s_1(t)\right)dt\right]$$

$$= E\left[\int_0^T s_0'(t)\left(s_0(t)-s_1(t)\right)\right] + E\left[\int_0^T N(t)\cdot\left(s_0(t)-s_1(t)\right)\right]$$

$$= \int_0^T s_0'(t)\cdot\left(s_0(t)-s_1(t)\right)dt + \int_0^T \underbrace{\cancel{E[N(t)]}}_{=0}\cdot\left(s_0(t)-s_1(t)\right)$$

$$= \boxed{E_0 - r_{01}} \qquad r_{01} - E_1$$

$$E_0 = \int_0^T s_0^2(t)\,dt \qquad r_{01} = \int_0^T s_0(t)\cdot s_1(t)\,dt$$

## Variance:

$$\mathrm{Var}[R \mid s_0(t)] = E\left[\left(R - E[R \mid s_0(t)]\right)^2 \Big| \, s_0(t)\right]$$

$$= E\left[\left(\underbrace{\int_0^T (s_0(t)+N(t))\cdot(s_0(t)-s_1(t))\,dt}_{=R} - \underbrace{\int_0^T s_0(t)\cdot(s_0(t)-s_1(t))}_{=E[R \mid s_0(t)]}\right)^2\right]$$

$$= E\left[\left(\int N(t)\cdot(s_0(t)-s_1(t))\,dt\right)^2\right]$$

$$= \int_0^T\!\!\int_0^T \underbrace{E[N(t)\cdot N(u)]}_{\frac{N_0}{2}\delta(t-u)} \cdot (s_0(t)-s_1(t))\cdot(s_0(u)-s_1(u))\,dt\,du$$

$$= \frac{N_0}{2}\cdot\int_0^T (s_0(u)-s_1(u))^2\,du$$

$$= \frac{N_0}{2}\cdot\int \left(s_0^2(t) - 2s_0(t)\cdot s_1(t) + s_1^2(t)\right)dt$$

$$= \boxed{\frac{N_0}{2}\cdot\left(E_0 + E_1 - 2r_{01}\right)}$$

with $E_1 = \int_0^T s_1^2(t)\,dt$

Similarly, when $s_1(t)$ was sent:

$$R \sim N\left(r_{01} - E_1,\ \frac{N_0}{2}(E_0 + E_1 - 2r_{01})\right)$$

## ② Find conditional prob. of error

a) $\Pr\{R > K \mid s_1(t)\} =$

$$\int_K^{\infty} f_{R|s_1}(r)\, dr =$$

$$f_{R|s_1} \leftarrow N\Big(\underbrace{r_{01} - E_1}_{\mu_1},\ \underbrace{\frac{N_0}{2}(E_0 + E_1 - 2r_{01})}_{\sigma^2}\Big)$$

$$\int_K^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{(r-\mu_1)^2}{\sigma^2}\right) dr =$$

$$z = \frac{r-\mu_1}{\sigma}: \quad \int_{\frac{K-\mu_1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \exp\left(-\frac{1}{2}z^2\right) dz \cdot \cancel{\sigma} =$$

$$Q\left(\frac{K-\mu_1}{\sigma}\right) = Q\left(\frac{K - r_{01} + E_1}{\sqrt{\frac{N_0}{2}(E_0 + E_1 - 2r_{01})}}\right)$$

b) $\Pr\{R < K \mid s_0(t)\}$

$$= \int_{-\infty}^{K} f_{R|s_0}(r)\, dr$$

$$f_{R|s_0} \leftarrow N\Big(\overbrace{E_0 - r_{01}}^{\mu_0},\ \overbrace{\frac{N_0}{2}(E_0 + E_1 - 2r_{01})}^{\sigma^2}\Big)$$

$$= \int_{-\infty}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \frac{(r-\mu_0)^2}{\sigma^2}\right) dr$$

$z = \frac{r-\mu_0}{\sigma}$ :

$$= \int_{-\infty}^{\frac{K-\mu_0}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= 1 - Q\left(\frac{K-\mu_0}{\sigma}\right) = Q\left(\frac{\mu_0 - K}{\sigma}\right)$$

$Q(-x) = 1 - Q(x)$

$$= Q\left(\frac{E_0 - r_{01} - K}{\sqrt{\frac{N_0}{2}(E_0 + E_1 - 2r_{01}}}\right)$$