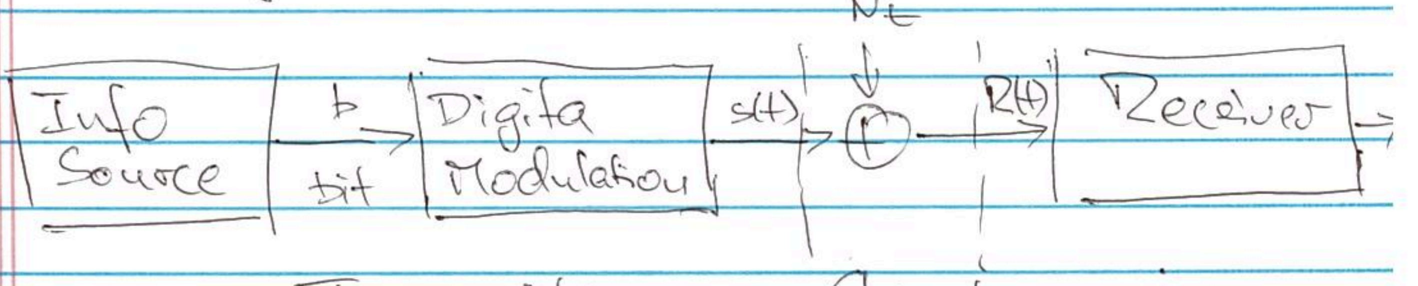
Finding P_e for a Simple Comms System



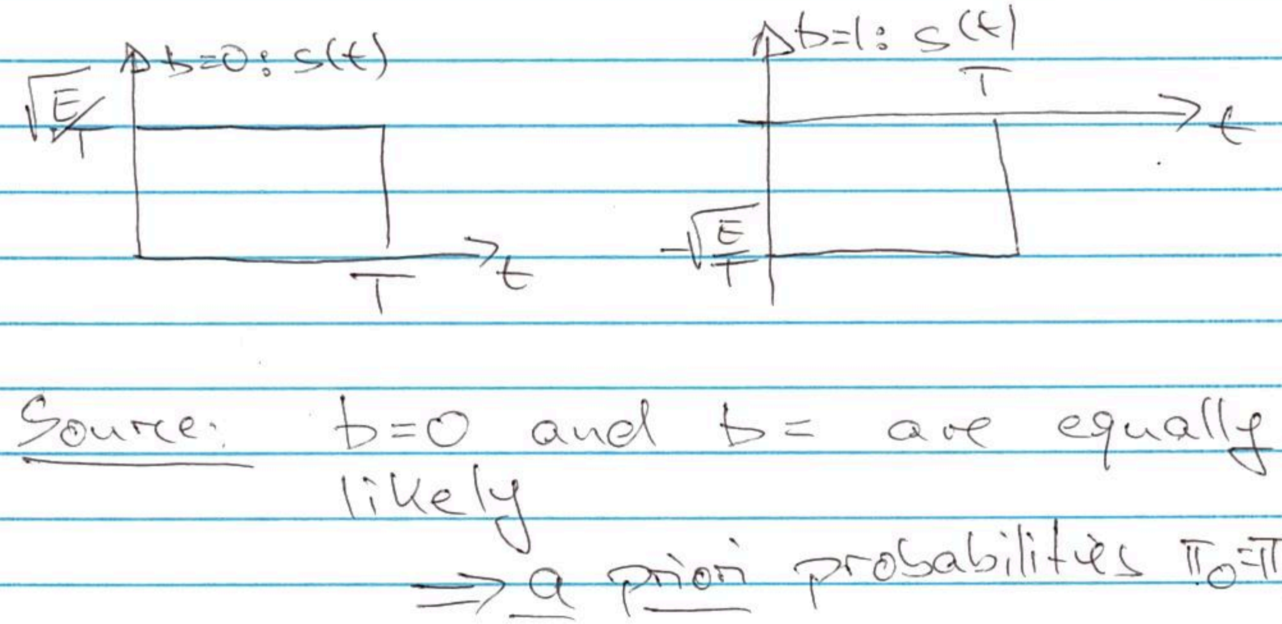
Transmitter   Channel (AWGN)

Example:

Modulation: BPSK signal

If $b=0$, then $s(t) = \sqrt{\frac{E}{T}}$ for $0 \le t \le T$

If $b=1$, then $s(t) = -\sqrt{\frac{E}{T}}$ for $0 \le t \le T$



Source: $b=0$ and $b=$ are equally likely

$\Rightarrow$ a prior probabilities $\pi_0 = \pi_1 = \frac{1}{2}$

Channel:

additive white Gaussian noise channel, spectral height $\frac{N_0}{2}$

$R_N(\tau) = \frac{N_0}{2}\delta(\tau)$ $\leftrightarrow$ $S_N(f) = \frac{N_0}{2}$ for all f

Receiver:

$$R(t) \longrightarrow \boxed{\int_0^T dt} \xrightarrow{R} \boxed{\underset{\hat{b}=1}{\overset{\hat{b}=0}{\gtrless}} 0} \longrightarrow \hat{b}$$

If $R > 0$, decide $b = 0$

If $R < 0$, decide $b = 1$

Want: Probability of Error

$$P_e = \Pr\{\hat{b} \neq b\}$$

$$= \Pr\{\hat{b} = 1 \mid b = 0\} \cdot \pi_0 + \Pr\{\hat{b} = 0 \mid b = 1\} \cdot \pi_1$$

$$= \Pr\{R < 0 \mid b = 0\} \cdot \pi_0 + \Pr\{R > 0 \mid b = 1\} \cdot \pi_1$$

Plan:

1.) Find the conditional pdf of R given $b = 0$ or $b = 1$

2.) From conditional pdfs, compute conditional error probabilities

$$\Pr\{R < 0 \mid b = 0\} \quad \text{or}$$

$$\Pr\{R > 0 \mid b = 1\}$$

1. <u>Find pdf of R given $b=\overset{1}{0}$</u>:

<u>Recall</u>: If $b=\overset{1}{0}$, then $s(t)=-\sqrt{\frac{E}{T}}$ for $0\le t<T$

Thus:

$$R(t) = -\sqrt{\frac{E}{T}} + N(t) \quad \text{for } 0\le t<T$$

$$\Rightarrow R = \int_0^T R(t)\,dt$$

$$= \int_0^T \left(-\sqrt{\frac{E}{T}} + N(t)\right) dt$$

$$= \int_0^T -\sqrt{\frac{E}{T}}\,dt + \int_0^T N(t)\,dt$$

$$= -\sqrt{E\cdot T} + \underbrace{\int_0^T N(t)\,dt}_{\text{this is Gaussian!}}$$

<u>Note</u>: R is Gaussian (conditional on $b=\overset{1}{0}$)

- $E[R\,|\,b=\overset{1}{0}] = -\sqrt{E\cdot T}$ $\left(\text{b/c } E\left[\int_0^T N(t)\,dt\right]=0\right)$

- $\mathrm{Var}[R\,|\,b=\overset{1}{0}] = E\left[\left(R - E[R|b=0]\right)^2 \,\middle|\, b=0\right]$

$$= E\left[\left(\underbrace{\cancel{-\sqrt{ET}} + \int_0^T N(t)\,dt}_{=R|b=0} \underbrace{\cancel{\overset{+}{-}\sqrt{ET}}}_{E[R|b=0]}\right)^2\right]$$

$$= E\left[\left(\int_0^T N(t)\,dt\right)^2\right]$$

$$= E\left[\int_0^T N(t)\,dt \int_0^T N(u)\,du\right]$$

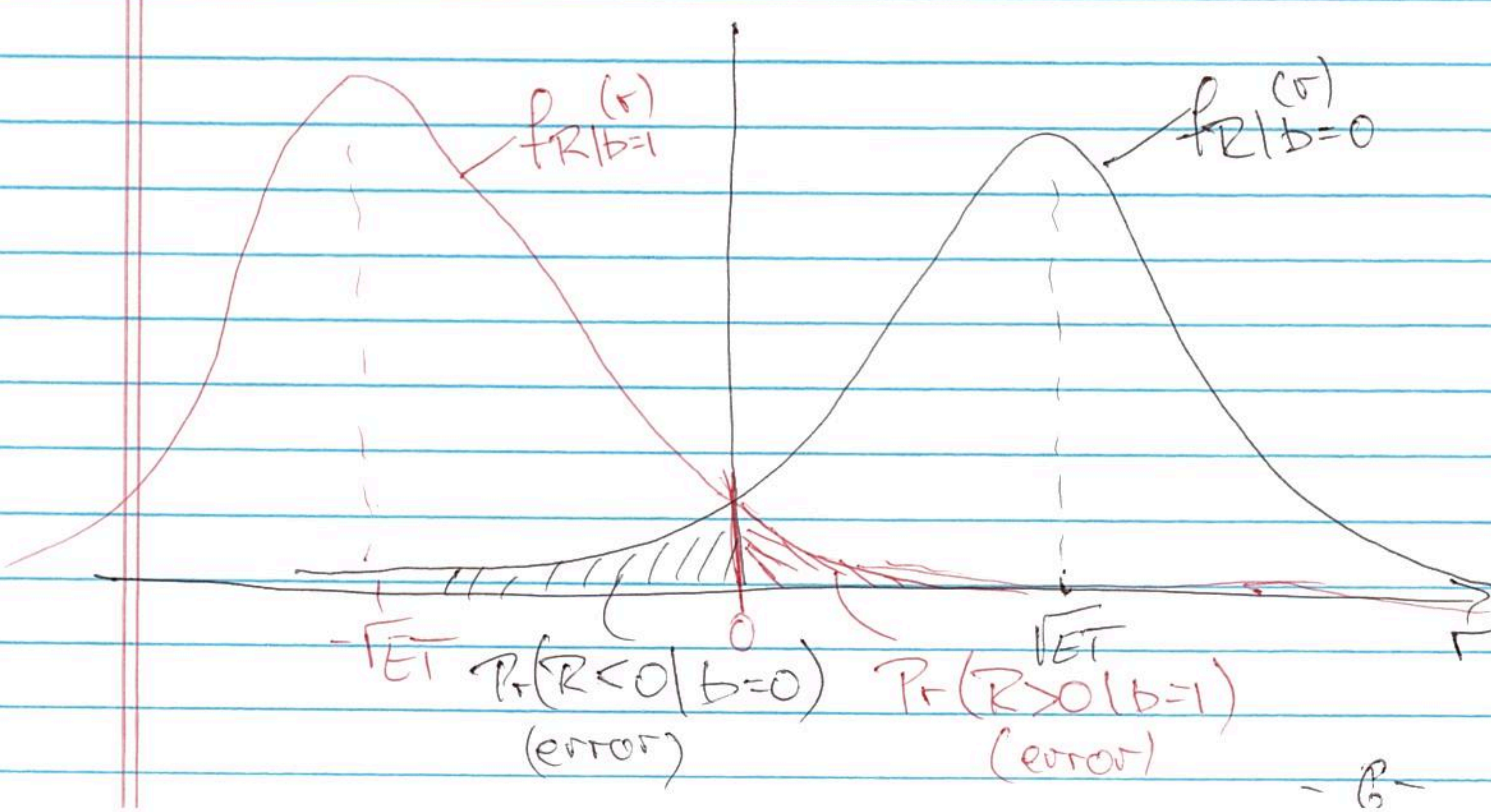
$$= \int_0^T\int_0^T \underbrace{E[N(t)\cdot N(u)]}_{\frac{N_0}{2}\delta(t-u)}\,dt\,du$$

$$= \int_0^T \frac{N_0}{2}\,du = \underline{\frac{N_0}{2}\cdot T}$$

If $b=0$: $R \sim N\left(\sqrt{ET}, \frac{N_0}{2}\cdot T\right)$

Similarly,:

If $b=1$: $R \sim N\left(-\sqrt{ET}, \frac{N_0}{2}\cdot T\right)$



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2. Find conditional probabilities of error:

$$\Pr\{R>0 \mid b=1\} = \int_0^{\infty} f_{R|b=1}(r)\, dr$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2} \cdot T}} \cdot \exp\left(-\frac{1}{2} \frac{(r-(-\sqrt{ET}))^2}{\frac{N_0}{2} \cdot T}\right) dr$$

Subst.: $z = \frac{r+\sqrt{ET}}{\sqrt{\frac{N_0}{2} \cdot T}}$

$$= \int_{\sqrt{\frac{ET}{\frac{N_0}{2}\cdot T}}}^{\infty} \frac{1}{\sqrt{2\pi \cancel{\frac{N_0}{2}T}}} \cdot \exp\left(-\frac{z^2}{2}\right) dz \cdot \cancel{\sqrt{\frac{N_0}{2} \cdot T}}$$

$$\sqrt{\frac{ET}{\frac{N_0}{2} \cdot \cancel{T}}} = \sqrt{\frac{2E}{N_0}}$$

$$= Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

by symmetry: $\Pr\{R<0 \mid b=0\} = Q\left(\sqrt{\frac{2E}{N_0}}\right)$

$$\Rightarrow P_e = \pi_0 \cdot \Pr\{R<0 \mid b=0\} + \pi_1 \cdot \Pr\{R>0 \mid b=1\}$$

$$= \frac{1}{2} \cdot Q\left(\sqrt{\frac{2E}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$Q(x)$ is monotonically decreasing

$\Rightarrow$ For small P_e, want large $\sqrt{\frac{2E}{N_0}}$!

The ratio $\frac{E}{N_0}$ is the ratio of

- energy used to send one bit
- Noise power spectral density

P_e decreases when $\frac{E}{N_0}$ increases

$\frac{E}{N_0}$ is called signal-to noise ratio (SNR)