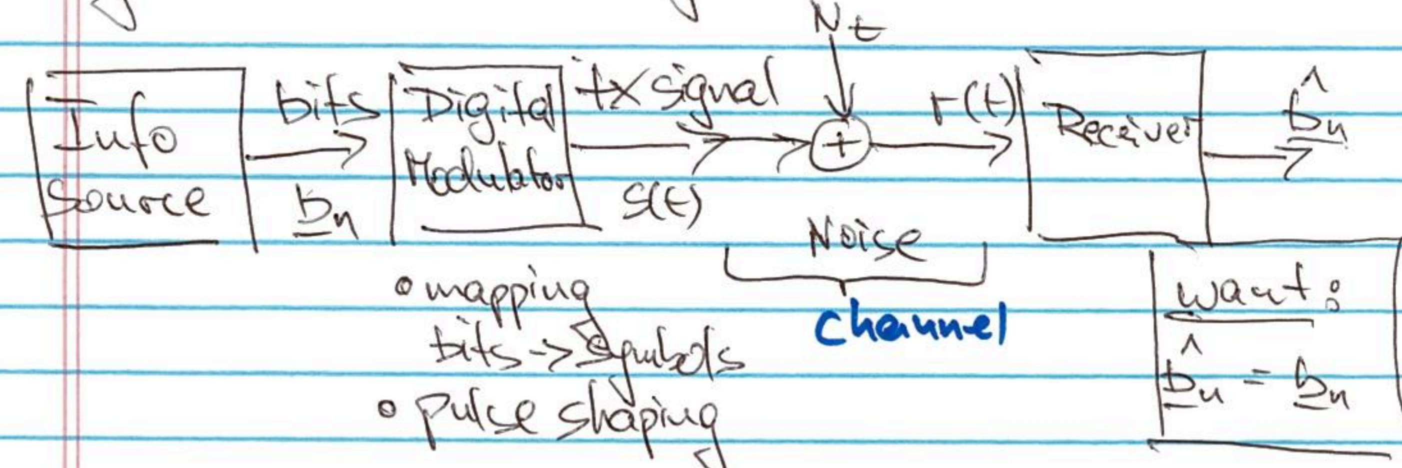
# Intro to Receivers for Digital Comms 10/29/18

## Digital Comms System



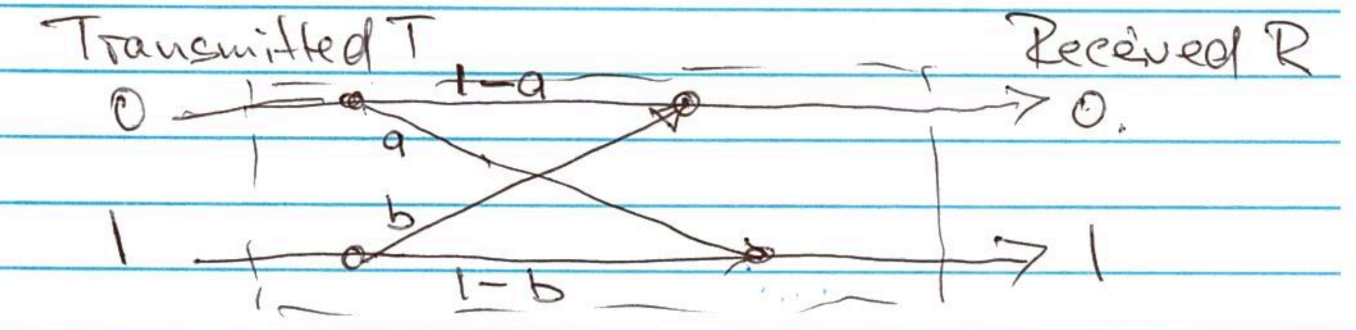

<u>Channel:</u> adds noise to the transmitted signal
(Additive Whit Gaussian Noise - AWGN)

<u>Receiver:</u> seeks to determine bits $\underline{b}_n$ from observing the <u>noisy</u> received signal $r(t)$.

<u>Metric:</u> Probability of error, i.e. probability that $\hat{b}_n \neq b_n$

Error probability (for binary, symmetric channel)

BSC:




BSC is characterized by transition probabilities:

– Conditional probalities

$$\Pr[R=r \mid T=t]$$

$$\text{E.g. } \Pr[R=1 \mid T=0] = a$$

Q: What is the probability of error?

To answer this question, we also need a priori probabilities $\pi_0, \pi_1$

$$\pi_0 = \Pr[T=0], \quad \pi_1 = \Pr[T=1]$$

(property of Info source

Usually: $\pi_0 = \pi_1 = \frac{1}{2}$

$$P_e = \Pr[\text{Error}]$$

$$= \Pr[R \neq T]$$

$$= \underbrace{\Pr[R=0 \mid T=1] \cdot \Pr[T=1]}_{= \Pr[R=0,\, T=1]} + \underbrace{\Pr[R=1 \mid T=0] \cdot \Pr[T=}_{= \Pr[R=1,\, T=0]}$$

$$= \pi_1 \cdot \Pr[R=0 \mid T=1] + \pi_0 \cdot \Pr[R=1 \mid T=0]$$

$$= \pi_1 \cdot b + \pi_0 \cdot a$$

- We will always compute $P_e$ this way
- Transition probabilities $P[R \mid T]$ depend on:
  - constellation
  - pulse shape
  - noise
  - received power
  - receiver structure
  
    ⋮

# Facts about Gaussian Random Variables

<u>Notation:</u> $X \sim N(\mu, \sigma^2)$

name of random variable

"distributed like"

Gaussian with mean: $\mu$ variance $\sigma^2$

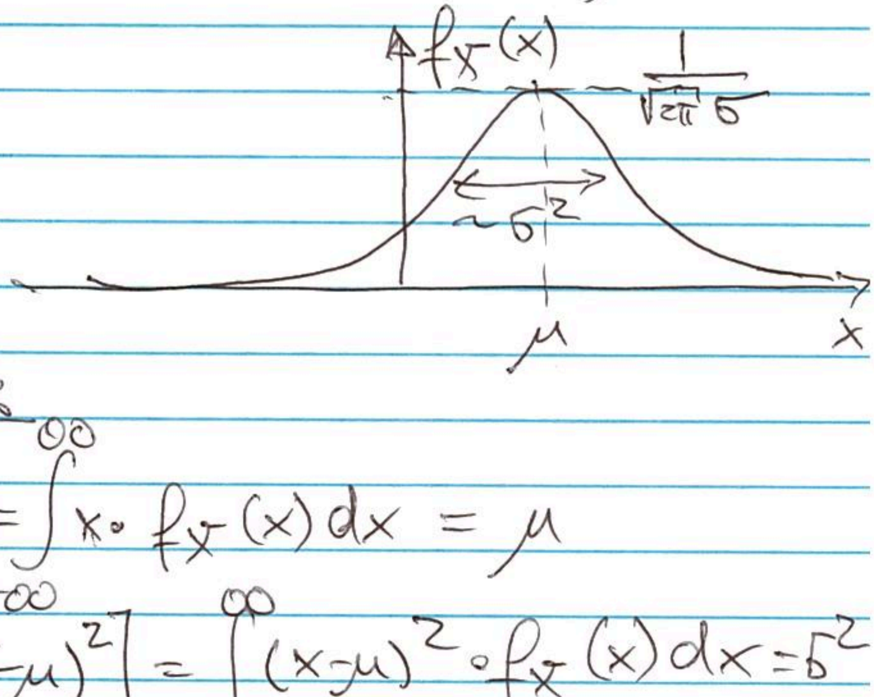
Then, $X$ has probability density function (pdf):

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

pdf

name of r.v.

"dummy" variable




<u>Two parameters:</u>

<u>Mean:</u> $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \mu$

<u>Variance:</u> $E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx = \sigma^2$

Cumulative distribution function (cdf)

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

no closed form solution

Often need:

$$\Pr\{X > \gamma\} = \int_{\gamma}^{\infty} f_X(x)\,dx$$

$$= \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Subst:

$z = \frac{x-\mu}{\sigma}$ ;

$dz = \frac{dx}{\sigma}$

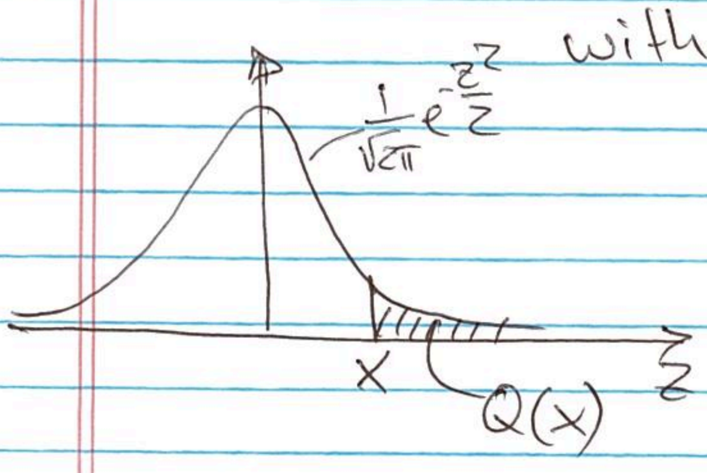
$$= \int_{\frac{\gamma-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \cdot \exp\left(-\frac{z^2}{2}\right) dz \cdot \cancel{\sigma}$$

$$= Q\left(\frac{\gamma-\mu}{\sigma}\right)$$

by definition

with $\boxed{Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz}$

$$= 1 - F_X(x)$$



## Awesome Property of Gaussians

The sum (more generally, a linear combination) of Gaussian random variables is itself a Gaussian random variable.

Example: Let $X_1, X_2, \ldots X_N$ be independent Gaussian random variables:

$$X_n \sim N(\mu, \sigma^2)$$

Form $Y = \sum_{n=1}^{N} X_n$

We know (from awesome property) that $Y$ is Gaussian.

$\Rightarrow$ need mean, variance

Mean: $E[Y] = E\left[\sum_{n=1}^{N} X_n\right]$

$$= \sum_{n=1}^{N} \underbrace{E[X_n]}_{=\mu} = \underline{\underline{N \cdot \mu}}$$

by linearity of $E[\cdot]$

Variance:

$$Var[Y] = E\left[(Y - E[Y])^2\right]$$

$$= E\left[\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \mu\right)^2\right]$$

$$= E\left[\left(\sum (x_n - \mu)\right)^2\right]$$

$$= E\left[\sum_n (x_n - \mu) \cdot \sum_m (x_m - \mu)\right]$$

$$= E\left[\sum_n \sum_m (x_n - \mu) \cdot (x_m - \mu)\right]$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} E\left[(x_n - \mu)(x_m - \mu)\right]$$

Q: what is $E[(x_n - \mu)(x_m - \mu)]$?

if $n = m$: $E[(x_n - \mu)^2] = Var[x_n] = \sigma^2$

if $n \neq m$: $E[(x_n - \mu)(x_m - \mu)]$

$$\underset{\text{indeped.}}{=} \cancel{E[(x_n - \mu)]}^{=0} \cdot \cancel{E[(x_m - \mu)]}^{=0} = 0$$

$$\Rightarrow \boxed{Var[Y] = N \cdot \sigma^2} \cancel{+ (N-1) \cdot N \cdot 0}$$