Exercise

Assume that the signal $x[n] = \exp(j2\pi\hat{f}n)$ is input to a 4-point averager.

1. Give a general expression for the output signal and identify the frequenchy response of the system.
2. Compute the output signals for the specific frequencies $\hat{f} = 0$, $\hat{f} = 1/4$, and $\hat{f} = 1/2$.

Lecture: The Frequency Response of LTI Systems

Introduction

- We have demonstrated that for linear, time-invariant systems
  - the output signal $y[n]$
  - is the convolution of the input signal $x[n]$ and the impulse response $h[n]$.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=0}^{M} h[k] \cdot x[n-k] \end{aligned}$$

- **Question:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi\hat{f}n + \phi)$.

Response to a Complex Exponential

- **Problem:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi\hat{f}n + \phi)$.
- Output $y[n]$ is convolution of input and impulse response

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=0}^{M} h[k] \cdot x[n-k] \\ &= \sum_{k=0}^{M} h[k] \cdot A\exp(j(2\pi\hat{f}(n-k) + \phi) \\ &= A\exp(j(2\pi\hat{f}n + \phi) \cdot \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi\hat{f}k) \\ &= A\exp(j(2\pi\hat{f}n + \phi) \cdot H(\hat{f}) \end{aligned}$$

- The term

$$H(\hat{f}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi\hat{f}k)$$

is called the Frequency Response of the system.

Interpreting the Frequency Response

The Frequency Response of an LTI system with impulse response $h[n]$ is

$$H(\hat{f}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi\hat{f}k)$$

- **Observations:**
  - The response of a LTI system to a complex exponential signal is a complex exponential signal of the same frequency.
    - Complex exponentials are eigenfunctions of LTI systems.
  - When $x[n] = A\exp(j(2\pi\hat{f}n + \phi))$, then $y[n] = x[n] \cdot H(\hat{f})$.
    - This is true only for exponential input signals, including complex exponentials!

Interpreting the Frequency Response

- **Observations:**
  - $H(\hat{f})$ is best interpreted in polar coordinates:

$$H(\hat{f}) = |H(\hat{f})| \cdot e^{j\angle H(\hat{f})}.$$

    - Then, for $x[n] = A\exp(j(2\pi\hat{f}n + \phi))$

$$\begin{aligned} y[n] &= x[n] \cdot H(\hat{f}) \\ &= A\exp(j(2\pi\hat{f}n + \phi)) \cdot |H(\hat{f})| \cdot e^{j\angle H(\hat{f})} \\ &= (A \cdot |H(\hat{f})|) \cdot \exp(j(2\pi\hat{f}n + \phi + \angle H(\hat{f}))) \end{aligned}$$

  - The amplitude of the resulting complex exponential is the product $A \cdot |H(\hat{f})|$.
    - Therefore, $|H(\hat{f})|$ is called the gain of the system.
  - The phase of the resulting complex exponential is the sum $\phi + \angle H(\hat{f})$.
    - $\angle H(\hat{f})$ is called the phase of the system.

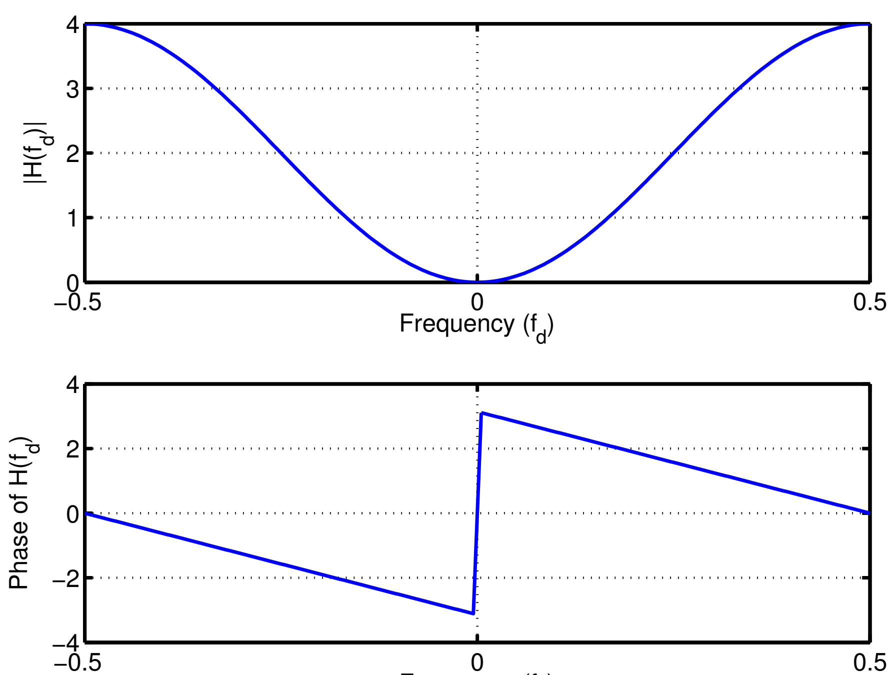
Example

- Let $h[n] = \{1, -2, 1\}$.
- Then,

$$\begin{aligned} H(\hat{f}) &= \sum_{k=0}^{2} h[k] \cdot \exp(-j2\pi\hat{f}k) \\ &= 1 - 2 \cdot \exp(-j2\pi\hat{f}) + 1 \cdot \exp(-j2\pi\hat{f}2) \\ &= \exp(-j2\pi\hat{f}) \cdot (\exp(j2\pi\hat{f}) - 2 + \exp(-j2\pi\hat{f})) \\ &= \exp(-j2\pi\hat{f}) \cdot (2\cos(2\pi\hat{f}) - 2). \end{aligned}$$

- Gain: $|H(\hat{f})| = |(2\cos(2\pi\hat{f}) - 2|$

Example



Example

- The filter with impulse response $h[n] = \{1, -2, 1\}$ is a high-pass filter.
  - It rejects sinusoids with frequencies near $\hat{f} = 0$,
  - and passes sinusoids with frequencies near $\hat{f} = \frac{1}{2}$
- Note how the function of this system is much easier to describe in terms of the frequency response $H(\hat{f})$ than in terms of the impulse response $h[n]$.
- **Question:** Find the output signal when input equals $x[n] = 2\exp(j2\pi 1/4n - \pi/2)$.
- **Solution:**

$$H(\frac{1}{4}) = \exp(-j2\pi\frac{1}{4}) \cdot (2\cos(2\pi\frac{1}{4}) - 2) = -2e^{-j\pi/2} = 2e^{j\pi/2}.$$

Thus,

$$y[n] = 2e^{j\pi/2} \cdot x[n] = 4\exp(j2\pi n/4).$$

Exercise

1. Find the Frequency Response $H(\hat{f})$ for the LTI system with impulse response $h[n] = \{1, -1, -1, 1\}$.
2. Find the output for the input signal $x[n] = 2\exp(j(2\pi n/3 - \pi/4))$.

Computing Frequency Response in MATLAB

```
function HH = FreqResp( hh, ff )
% FreqResp - compute frequency response of LTI system
%
% inputs:
%     hh - vector of impulse repsonse coefficients
%     ff - vector of frequencies at which to evaluate frequency respon
%
% output:
%     HH - frequency response at frequencies in ff.
%
% Syntax:
%     HH = FreqResp( hh, ff )

HH = zeros( size(ff) );
for kk = 1:length(hh)
    HH = HH + hh(kk)*exp(-j*2*pi*(kk-1)*ff);
end
```


Part VII

Appendix: Complex Numbers and Complex Algebra