Lecture: Linear, Time-Invariant Systems

Introduction

- We have introduced systems as devices that process an input signal $x[n]$ to produce an output signal $y[n]$.
- **Example Systems:**
  - **Squarer:** $y[n] = (x[n])^2$
  - **Modulator:** $y[n] = x[n] \cdot \cos(2\pi \hat{f} n)$, with $0 < \hat{f} \leq \frac{1}{2}$.
  - **FIR Filter:**

    $$y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k].$$

    Recall that $h[k]$ is the impulse response of the filter and that the above operation is called convolution of $h[n]$ and $x[n]$.
- **Objective:** Define important characteristics of systems and determine which systems possess these characteristics.

Causal Systems

- **Definition:** A system is called causal when it uses only the present and past samples of the input signal to compute the present value of the output signal.
- Causality is usually easy to determine from the system equation:
  - The output $y[n]$ must depend only on input samples $x[n], x[n-1], x[n-2], \ldots$.
  - Input samples $x[n+1], x[n+2], \ldots$ must not be used to find $y[n]$.
- **Examples:**
  - All three systems on the previous slide are causal.
  - The following system is non-causal:

$$y[n] = \frac{1}{3}\sum_{k=-1}^{1} x[n-k] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]).$$

Linear Systems

- The following test procedure defines linearity and shows how one can determine if a system is linear:
  1. **Reference Signals:** For $i = 1, 2$, pass input signal $x_i[n]$ through the system to obtain output $y_i[n]$.
  2. **Linear Combination:** Form a new signal $x[n]$ from the linear combination of $x_1[n]$ and $x_2[n]$:

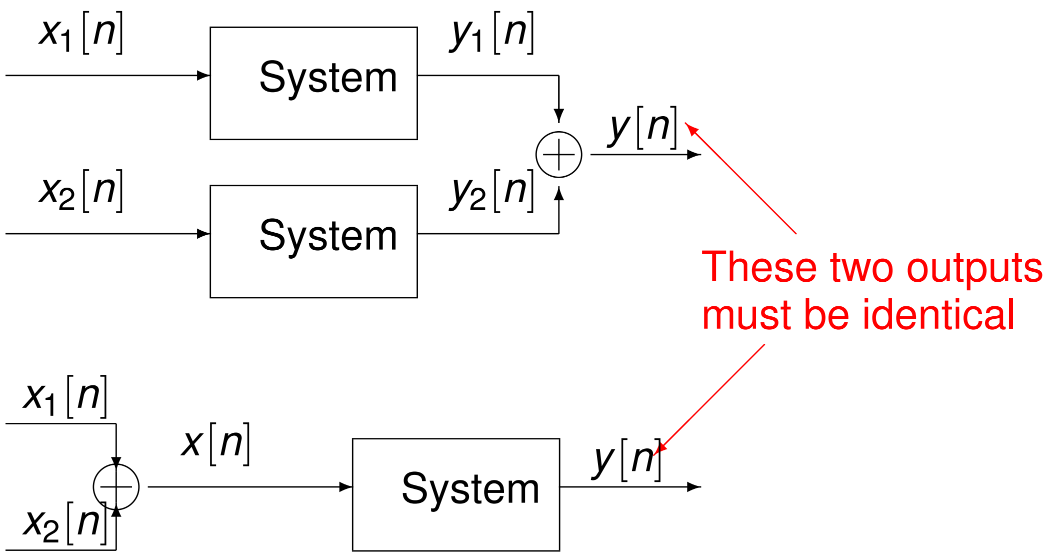
     $$x[n] = x_1[n] + x_2[n].$$

     Then, Pass signal $x[n]$ through the system and obtain $y[n]$.
  3. **Check:** The system is linear if

     $$y[n] = y_1[n] + y_2[n]$$

- The above must hold for **all** inputs $x_1[n]$ and $x_2[n]$.
- For a linear system, the superposition principle holds.

Illustration



Example: Squarer

- **Squarer:** $y[n] = (x[n])^2$
  1. **References:** $y_i[n] = (x_i[n])^2$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and

$$\begin{aligned} y[n] &= (x[n])^2 = (x_1[n] + x_2[n])^2 \\ &= (x_1[n])^2 + (x_2[n])^2 + 2x_1[n]x_2[n]. \end{aligned}$$

  3. **Check:**

$$y[n] \neq y_1[n] + y_2[n] = (x_1[n])^2 + (x_2[n])^2.$$

- **Conclusion:** not linear.

Example: Modulator

- **Modulator:** $y[n] = x[n] \cdot \cos(2\pi \hat{f} n)$
  1. **References:** $y_i[n] = x_i[n] \cdot \cos(2\pi \hat{f} n)$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and

$$\begin{aligned} y[n] &= x[n] \cdot \cos(2\pi \hat{f} n) \\ &= (x_1[n] + x_2[n]) \cdot \cos(2\pi \hat{f} n). \end{aligned}$$

  3. **Check:**

$$y[n] = y_1[n] + y_2[n] = x_1[n] \cdot \cos(2\pi \hat{f} n) + x_2[n] \cdot \cos(2\pi \hat{f} n).$$

- **Conclusion:** linear.

Example: FIR Filter

- **FIR Filter:** $y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k]$
  1. **References:** $y_i[n] = \sum_{k=0}^{M} h[k] \cdot x_i[n-k]$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and

$$y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k] = \sum_{k=0}^{M} h[k] \cdot (x_1[n-k] + x_2[n-k]).$$

  3. **Check:**

$$y[n] = y_1[n] + y_2[n] = \sum_{k=0}^{M} h[k] \cdot x_1[n-k] + \sum_{k=0}^{M} h[k] \cdot x_2[n-k].$$

- **Conclusion:** linear.

Time-invariance

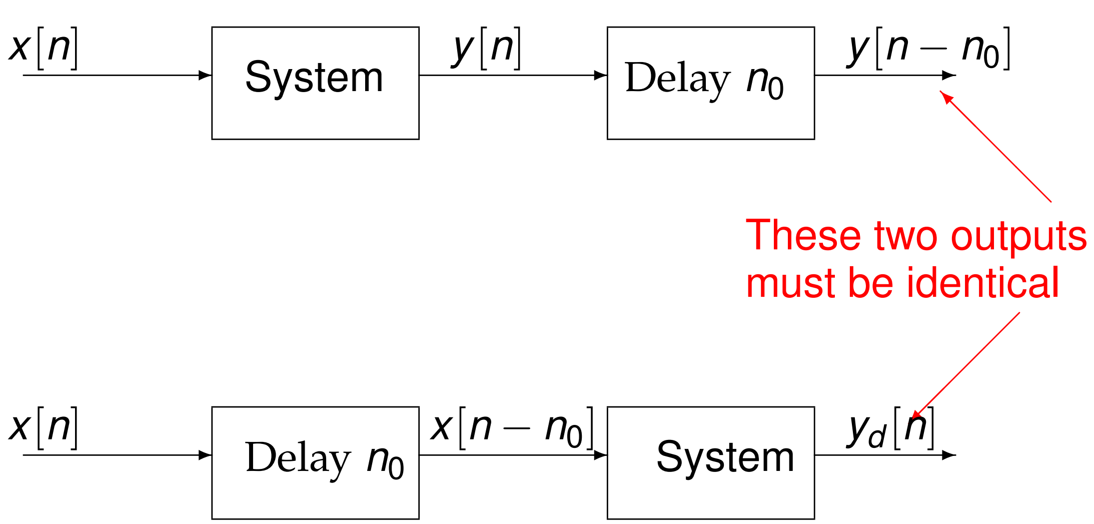
- The following test procedure defines time-invariance and shows how one can determine if a system is time-invariant:

  1. **Reference:** Pass input signal $x[n]$ through the system to obtain output $y[n]$.
  2. **Delayed Input:** Form the delayed signal $x_d[n] = x[n - n_0]$. Then, Pass signal $x_d[n]$ through the system and obtain $y_d[n]$.
  3. **Check:** The system is time-invariant if

$$y[n - n_0] = y_d[n]$$

- The above must hold for **all** inputs $x[n]$ and all delays n_0.
- **Interpretation:** A time-invariant system does not change, over time, the way it processes the input signal.

Illustration



Example: Squarer

- **Squarer:** $y[n] = (x[n])^2$
  1. **Reference:** $y[n] = (x[n])^2$.
  2. **Delayed Input:** $x_d[n] = x[n - n_0]$ and
  $$y_d[n] = (x_d[n])^2 = (x[n - n_0])^2.$$
  3. **Check:**
  $$y[n - n_0] = (x[n - n_0])^2 = y_d[n].$$
- **Conclusion:** time-invariant.

Example: Modulator

- **Modulator:** $y[n] = x[n] \cdot \cos(2\pi\hat{f}n)$.
  1. **Reference:** $y[n] = x[n] \cdot \cos(2\pi\hat{f}n)$.
  2. **Delayed Input:** $x_d[n] = x[n - n_0]$ and
  $$y_d[n] = x_d[n] \cdot \cos(2\pi\hat{f}n) = x[n - n_0] \cdot \cos(2\pi\hat{f}n).$$
  3. **Check:**
  $$y[n - n_0] = x[n - n_0] \cdot \cos(2\pi\hat{f}(n - n_0)) \neq y_d[n].$$
- **Conclusion:** not time-invariant.

Example: Modulator

- Alternatively, to show that the modulator is **not** time-invariant, we construct a counter-example.
- Let $x[n] = \{0, 1, 2, 3, \ldots\}$, i.e., $x[n] = n$, for $n \geq 0$.
- Also, let $\hat{f} = \frac{1}{2}$, so that

$$\cos(2\pi\hat{f}n) = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases}$$

- Then, $y[n] = x[n] \cdot \cos(2\pi\hat{f}n) = \{0, -1, 2, -3, \ldots\}$.
- With $n_0 = 1$, $x_d[n] = x[n-1] = \{0, 0, 1, 2, 3, \ldots\}$, we get $y_d[n] = \{0, 0, 1, -2, 3, \ldots\}$.
- Clearly, $y_d[n] \neq y[n-1]$.
- not time-invariant

Example: FIR Filter

- **Reference:** $y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k]$.
- **Delayed Input:** $x_d[n] = x[n - n_0]$, and

$$y_d[n] = \sum_{k=0}^{M} h[k] \cdot x_d[n-k] = \sum_{k=0}^{M} h[k] \cdot x[n - n_0 - k].$$

- **Check:**

$$y[n - n_0] = \sum_{k=0}^{M} h[k] \cdot x[n - n_0 - k] = y_d[n]$$

- time-invariant

Exercise

- Let $u[n]$ be the unit-step sequence (i.e., $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$, otherwise).
- The system is a 3-point averager:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]).$$

1. Find the output $y_1[n]$ when the input $x_1[n] = u[n]$.
2. Find the output $y_2[n]$ when the input $x_2[n] = u[n-2]$.
3. Find the output $y_3[n]$ when the input $x[n] = u[n] - u[n-2]$.
4. Use linearity and time-invariance to find $y_2[n]$ and $y_3[n]$ without convolution.

Part VI

Appendix: Complex Numbers and Complex Algebra