Finding the Fundamental Frequency

- Often one is given a set of frequencies $f_1, f_2, \ldots, f_N$ and is required to find the fundamental frequency f_0.
- Specifically, this means one must find a frequency f_0 and integers $n_1, n_2, \ldots, n_N$ such that all of the following equations are met:

$$\begin{aligned} f_1 &= n_1 \cdot f_0 \\ f_2 &= n_2 \cdot f_0 \\ &\vdots \\ f_N &= n_N \cdot f_0 \end{aligned}$$

- Note that there isn't always a solution to the above problem.
  - However, if all frequencies are integers a solution exists.
  - Even if all frequencies are rational a solution exists.

Example

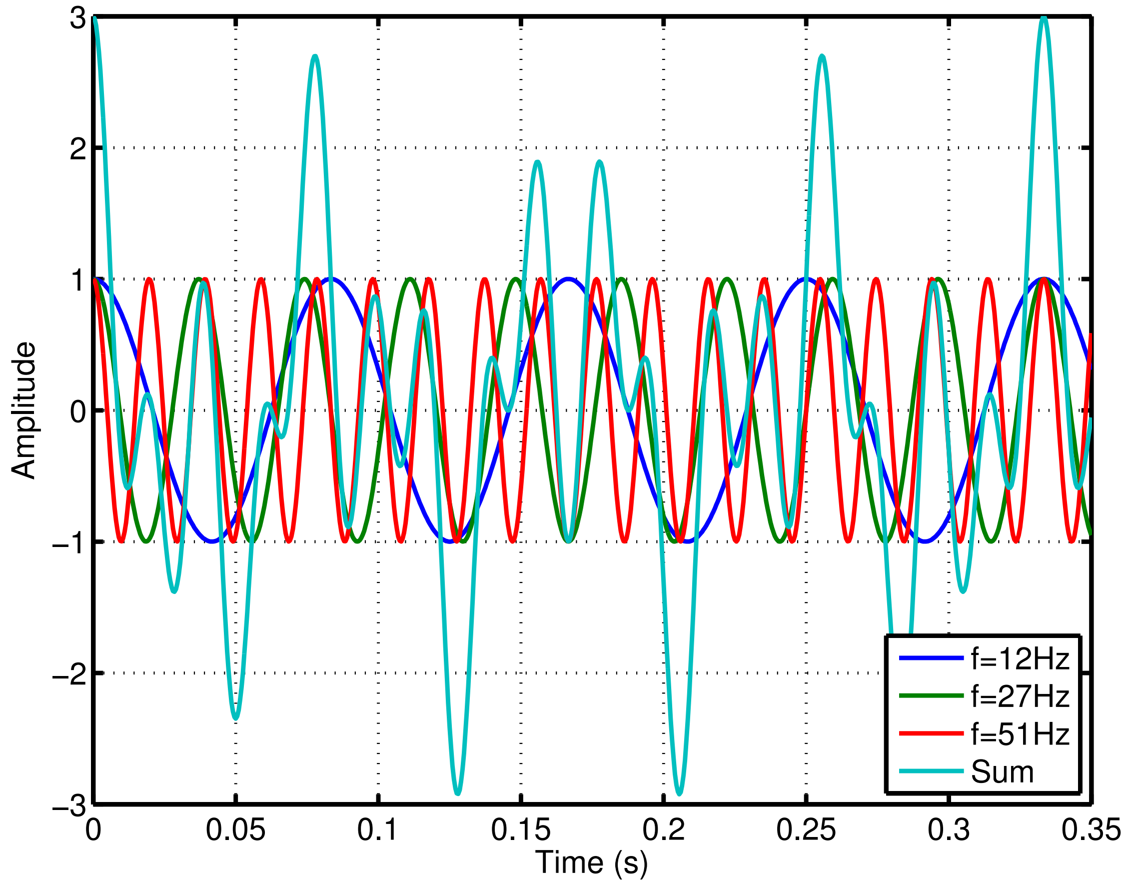
- Find the fundamental frequency for the set of frequencies $f_1 = 12$, $f_2 = 27$, $f_3 = 51$.
- Set up the equations:

$$\begin{aligned} 12 &= n_1 \cdot f_0 \\ 27 &= n_2 \cdot f_0 \\ 51 &= n_3 \cdot f_0 \end{aligned}$$

- Try the solution $n_1 = 1$; this would imply $f_0 = 12$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 2$; this would imply $f_0 = 6$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 3$; this would imply $f_0 = 4$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 4$; this would imply $f_0 = 3$. This **can** satisfy the other two equations with $n_2 = 9$ and $n_3 = 17$.

Example

- Note that the three sinusoids complete a cycle at the same time at $T_0 = 1/f_0 = 1/3s$.



A Few Things to Note

- Note that the fundamental frequency f_0 that we determined is the greatest common divisor (gcd) of the original frequencies.
  - $f_0 = 3$ is the gcd of $f_1 = 12$, $f_2 = 27$, and $f_3 = 51$.
- The integers n_i are the number of full periods (cycles) the sinusoid of freqency f_i completes in the fundamental period $T_0 = 1/f_0$.
  - For example, $n_1 = f_1 \cdot T_0 = f_1 \cdot 1/f_0 = 4$.
  - The sinusoid of frequency f_1 completes $n_1 = 4$ cycles during the period T_0.

Exercise

- Find the fundamental frequency for the set of frequencies $f_1 = 2$, $f_2 = 3.5$, $f_3 = 5$.

Fourier Series

- We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.
- One can turn this statement around and arrive at a very important result:

  Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.

- The resulting sum is called the Fourier Series of the signal.
- Put differently, a periodic signal can always be written in the form

$$\begin{aligned} x(t) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i f_0 t + \phi_i) \\ &= X_0 + \sum_{i=1}^{N} X_i e^{j2\pi i f_0 t} + X_i^* e^{-j2\pi i f_0 t} \end{aligned}$$

with $X_0 = A_0$ and $X_i = \frac{A_i}{2} e^{j\phi_i}$.

Fourier Series

- For a periodic signal the complex amplitudes X_i can be computed using a (relatively) simple formula.
- Specifically, for a periodic signal $x(t)$ with fundamental period T_0 the complex amplitudes X_i are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi it/T_0} dt.$$

  - Note that the integral above can be evaluated over any interval of length T_0.

Example: Square Wave

- A square wave signal is periodic and between $t - 0$ and $t = T_0$ it equals

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

- From the Fourier Series expansion it follows that $x(t)$ can be written as

$$x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$

25-Term Approximation to Square Wave

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$