# ECE 201: Introduction to Signal Analysis

Dr. B.-P. Paris
Dept. Electrical and Comp. Engineering
George Mason University

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# Part I

## Introduction

# Lecture: Introduction





## Learning Objectives

- Intro to Electrical Engineering via Digital Signal Processing.
- Develop initial understanding of Signals and Systems.
- Learn MATLAB
- Note: Math is not very hard - just algebra.

## DSP - Digital Signal Processing

**Digital:** processing via computers and digital hardware
we will use PC's.

**Signal:** Principally signals are just functions of time
- Entertainment/music
- Communications
- Medical, ...

**Processing:** analysis and transformation of signals
we will use MATLAB





## Outline of Topics

- Sinusoidal Signals
- Time and Frequency representation of signals
- Sampling
- Filtering
- Spectrum Analysis

- MATLAB
  - Lectures
  - Labs
  - Homework

## Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expresssed as sums of sinusoids (Fourier Theory)
- Bridge to frequency domain.
- Sinusoids are *special signals* for linear filters (eigenfunctions).
- Manipulating sinusoids is much easier with the help of complex numbers.





## Time and Frequency

- Closely related via sinusoids.
- Provide two different perspectives on signals.
- Many operations are easier to understand in frequency domain.

## Sampling

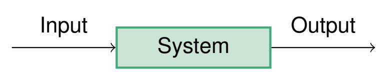
- Conversion from continuous time to discrete time.
- Required for Digital Signal Processing.
- Converts a signal to a sequence of numbers (samples).
- Straightforward operation
  - with a few *strange* effects.





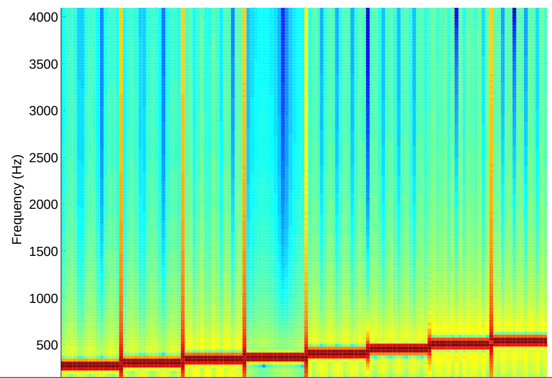
## Filtering

- A simple, but powerful, class of operations on signals.
- Filtering transforms an *input signal* into a more suitable *output signal*.
- Often best understood in frequency domain.



## Spectrum Analysis

- Analyze a given signal to find which frequencies it contains.
- Fourier Transform and fast Fourier Transform
- Spectrogram








## Relationship to other ECE Courses

- Next steps after ECE 201:
  - ECE 220: Signals and Systems
  - ECE 280: Circuits
- Core courses in controls and communications:
  - ECE 421: Controls
  - ECE 460: Communications
- Electives:
  - ECE 410: DSP
  - ECE 450: Robotics
  - ECE 463: Digital Comms
  - ECE 464: Filter Design

# Part II

## Sinusoids, Complex Numbers, and Complex Exponentials





## Lecture: Introduction to Sinusoids

## The Formula for Sinusoidal Signals

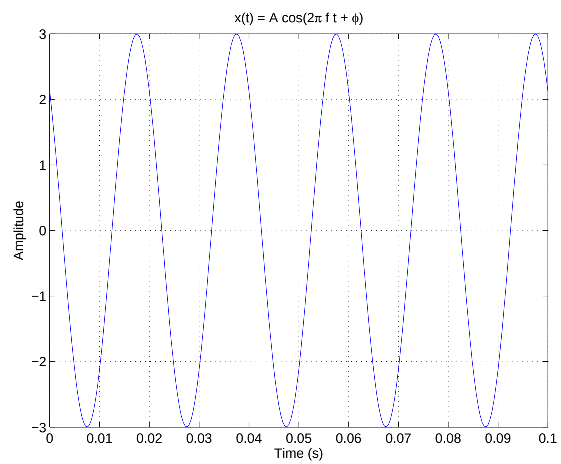
- The general formula for a sinusoidal signal is

$$x(t) = A \cdot \cos(2\pi f t + \phi).$$

- $A$, $f$, and $\phi$ are parameters that characterize the sinusoidal signal.
  - $A$ - Amplitude: determines the height of the sinusoid.
  - $f$ - Frequency: determines the number of cycles per second.
  - $\phi$ - Phase: determines the horizontal location of the sinusoid.








- The formula for this sinusoid is:

$$x(t) = 3 \cdot \cos(2\pi \cdot 50 \cdot t + \pi/4).$$

## The Significance of Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expresssed as sums of sinusoids (Fourier Theory)
  - Provides bridge to frequency domain.
- Sinusoids are *special signals* for linear filters (eigenfunctions).
- Sinusoids occur naturally in many situations.
  - They are solutions of differential equations of the form

$$\frac{d^2x(t)}{dt^2} + ax(t) = 0.$$

- Much more on these points as we proceed.





## Background: The cosine function

- The properties of sinusoidal signals stem from the properties of the cosine function:
  - **Periodicity:** $\cos(x + 2\pi) = \cos(x)$
  - **Eveness:** $\cos(-x) = \cos(x)$
  - **Ones** of cosine: $\cos(2\pi k) = 1$, for all integers $k$.
  - **Minus ones** of cosine: $\cos(\pi(2k + 1)) = -1$, for all integers $k$.
  - **Zeros** of cosine: $\cos(\frac{\pi}{2}(2k + 1)) = 0$, for all integers $k$.
  - Relationship to **sine function:** $\sin(x) = \cos(x - \pi/2)$ and $\cos(x) = \sin(x + \pi/2)$.

## Amplitude

- The amplitude $A$ is a *scaling factor*.
- It determines how large the signal is.
- Specifically, the sinusoid oscillates between $+A$ and $-A$.






## Frequency and Period

- Sinusoids are periodic signals.
- The frequency $f$ indicates how many times the sinusoid repeats per second.
- The duration of each cycle is called the period of the sinusoid.
  It is denoted by $T$.
- The relationship between frequency and period is

$$f = \frac{1}{T} \text{ and } T = \frac{1}{f}.$$

## Phase and Delay

- The phase $\phi$ causes a sinusoid to be shifted sideways.
- A sinusoid with phase $\phi = 0$ has a maximum at $t = 0$.
- A sinusoid that has a maximum at $t = \tau$ can be written as

$$x(t) = A \cdot \cos(2\pi f(t - \tau)).$$

- Expanding the argument of the cosine leads to

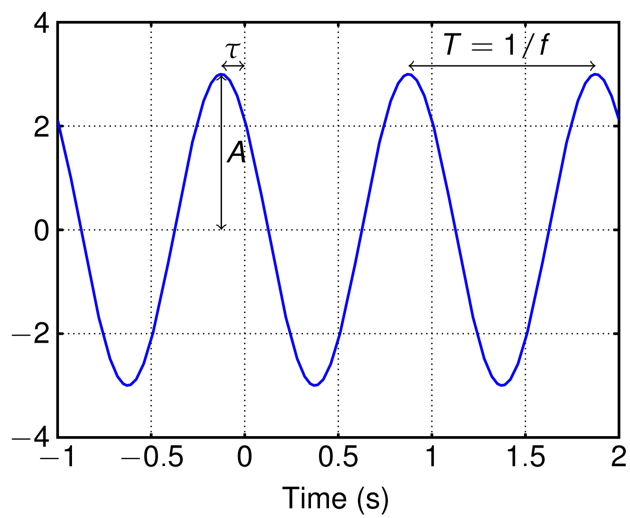
$$x(t) = A \cdot \cos(2\pi ft - 2\pi f\tau).$$

- Comparing to the general formula for a sinusoid reveals

$$\phi = -2\pi f\tau \text{ and } \tau = \frac{-\phi}{2\pi f}.$$







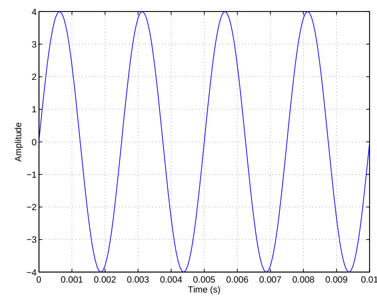
## Exercise

1. Plot the sinusoid

$$x(t) = 2\cos(2\pi \cdot 10 \cdot t + \pi/2)$$

   between $t = -0.1$ and $t = 0.2$.

2. Find the equation for the sinusoid in the following plot








## Vectors and Matrices

- MATLAB is specialized to work with vectors and matrices.
- Most MATLAB commands take vectors or matrices as arguments and perform looping operations automatically.
- Creating vectors in MATLAB:

  directly:

```
x = [ 1, 2, 3 ];
```

  using the increment (:) operator:

```
x = 1:2:10;
```

  produces a vector with elements `[1, 3, 5, 7, 9]`.

  using MATLAB commands For example, to read a .wav file

```
[ x, fs] = wavread('music.wav');
```

# Plot a Sinusoid

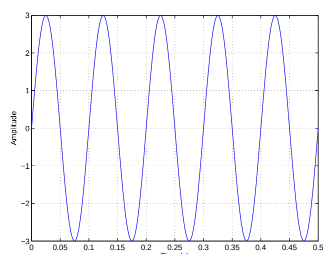
```
%% parameters
A   = 3;
f   = 50;
phi = pi/4;

fs  = 50*f;

%% generate signal
% 5 cycles with 50 samples per cycle
tt = 0 : 1/fs : 5/f;
xx = A*cos(2*pi*f*tt + phi);

%% plot
plot(tt,xx)
xlabel( 'Time␣(s)' )    % labels for x and y axis
ylabel( 'Amplitude' )
title( 'x(t)␣=␣A␣cos(2\pi␣f␣t␣+␣\phi)')
```





# Exercise

- The sinusoid below has frequency $f = 10$ Hz.
- Three of its maxima are at the the following locations $\tau_1 = -0.075\,\text{s}$, $\tau_2 = 0.025\,\text{s}$, $\tau_3 = 0.125\,\text{s}$
- Use each of these three delays to compute a value for the phase $\phi$ via the relationship $\phi_i = -2\pi f \tau_i$.
- What is the relationship between the phase values $\phi_i$ you obtain?



# Lecture: Adding Sinusoids of the Same Frequency

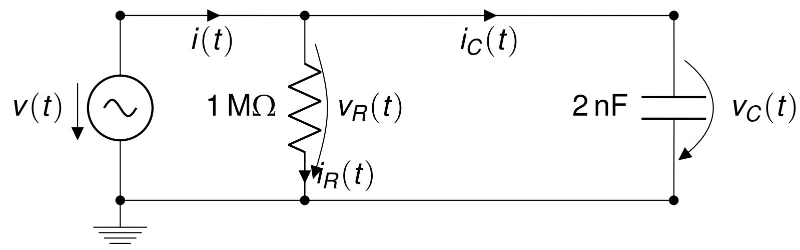




## Adding Sinusoids

- Adding sinusoids of the same frequency is a problem that arises regularly in
  - circuit analysis
  - linear, time-invariant systems, e.g., filters
  - and many other domains
- We will see that adding sinusoids is much easier with complex exponentials
  - Today, we will do it the hard way — with trigonometry

## A Circuits Example




- For $v(t) = 1\,\text{V} \cdot \cos(2\pi 1\,\text{kHz} \cdot t)$, find the current $i(t)$.



Sinusoidal Signals
Sums of Sinusoids
Complex Exponential Signals

## Setting up the Problem

- Resistor: $i_R(t) = \frac{v_R(t)}{R}$
- Capacitor: $i_C(t) = C\frac{dv_C(t)}{dt}$
- Kirchhoff's current law: $i(t) = i_R(t) + i_C(t)$
- Kirchhoff's voltage law: $v(t) = v_R(t) = v_C(t)$
- Therefore,

$$\begin{aligned}
i(t) &= \frac{v(t)}{R} + C \cdot \frac{dv(t)}{dt} \\
&= \frac{1\,\text{V}}{1\,\text{M}\Omega} \cos(2\pi 1\,\text{kHz} \cdot t) - 2\pi \cdot 1\,\text{kHz} \cdot 2\,\text{nF} \cdot \sin(2\pi 1\,\text{kHz} \cdot t) \\
&= 1\,\mu\text{A} \cos(2\pi 1\,\text{kHz} \cdot t) - 4\pi\,\mu\text{A} \sin(2\pi 1\,\text{kHz} \cdot \text{t})
\end{aligned}$$

## Simplifying $i(t)$

- Can we write

$$i(t) = 1\,\mu\text{A}\cos(2\pi 1\,\text{kHz}\cdot t) - 4\pi\,\mu\text{A}\sin(2\pi 1\,\text{kHz}\cdot \text{t})$$

  as a single sinusoid?
- Specifically, can we express it in the standard form

$$i(t) = I\cos(2\pi ft + \phi)$$

  and, if so, what are $I$, $f$, and $\phi$?





## Solution

- Use the trig identity
  - $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

  to change $i(t) = I\cos(2\pi ft + \phi)$ to

$$i(t) = I\cdot\cos(\phi)\cos(2\pi ft) - I\cdot\sin(\phi)\sin(2\pi ft)$$

- Compare to

$$i(t) = 1\,\mu\text{A}\cos(2\pi 1\,\text{kHz}\cdot t) - 4\pi\,\mu\text{A}\sin(2\pi 1\,\text{kHz}\cdot \text{t})$$

- Conclude:
  - $f = 1\,\text{kHz}$ - no change in frequency!
  - $I\cdot\cos(\phi) = 1\,\mu\text{A}$ and $I\cdot\sin(\phi) = 4\pi\,\mu\text{A}$.

## Solution

- We still must find $I$ and $\phi$ from
  - $I \cdot \cos(\phi) = 1\,\mu\text{A}$ and $I \cdot \sin(\phi) = 4\pi\,\mu\text{A}$.
- We can find $I$ from

$$\begin{array}{ccccc} I^2 \cdot \cos^2(\phi) & + & I^2 \cdot \sin^2(\phi) & = & I^2 \\ (1\,\mu\text{A})^2 & + & (4\pi\,\mu\text{A})^2 & \approx & (12.6\,\mu\text{A})^2 \end{array}$$

- Thus, $I = 12.6\,\mu\text{A}$.
- Also,

$$\frac{I \cdot \sin(\phi)}{I \cdot \cos(\phi)} = \tan(\phi) = \frac{4\pi}{1}.$$

- Hence, $\phi \approx 0.47 \cdot \pi \approx 85^o$.
- And, $i(t) \approx 12.6\,\mu\text{A} \cos(2\pi 1\,\text{kHz} \cdot t + 0.47 \cdot \pi)$.





## Exercise

- Express

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2)$$

  in the form $A \cdot \cos(2\pi f t + \phi)$.
- Answer: $x(t) \approx 5 \cos(2\pi f t + 53^o)$

## Solution to Exercise

- Express
$$x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi/2)$$
in the form $A \cdot \cos(2\pi ft + \phi)$.
- **Solution:** Use trig identity
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ on second term.
- This leads to
$$\begin{aligned} x(t) &= 3 \cdot \cos(2\pi ft) + \\ &\quad 4 \cdot \cos(2\pi ft)\cos(\pi/2) - 4 \cdot \sin(2\pi ft)\sin(\pi/2) \\ &= 3 \cdot \cos(2\pi ft) - 4 \cdot \sin(2\pi ft). \end{aligned}$$
- Compare to what we want:
$$\begin{aligned} x(t) &= A \cdot \cos(2\pi ft + \phi) \\ &= A \cdot \cos(\phi)\cos(2\pi ft) - A \cdot \sin(\phi)\sin(2\pi ft) \end{aligned}$$



Sinusoidal Signals | Sums of Sinusoids | Complex Exponential Signals

## Solution cont'd

- We can conclude that $A$ and $\phi$ must satisfy
$$A \cdot \cos(\phi) = 3 \text{ and } A \cdot \sin(\phi) = 4.$$
- We can find $A$ from
$$\begin{array}{ccccc} A^2 \cdot \cos^2(\phi) & + & A^2 \cdot \sin^2(\phi) & = & A^2 \\ 9 & + & 16 & = & 25 \end{array}$$
- Thus, $A = 5$.
- Also,
$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{4}{3}.$$
- Hence, $\phi \approx 53^o$ ($\frac{53}{180}\pi$).
- And, $x(t) = 5\cos(2\pi ft + 53^o)$.

## Summary

- Adding sinusoids of the same frequency is a problem that is frequently encountered in Electrical Engineering.
  - We noticed that the frequency of the sum of sinusoids is the same as the frequency of the sinusoids that we added.
- Such problems can be solved using trigonometric identities.
  - but, that is very tedious.
- We will see that sums of sinusoids are much easier to compute using complex algebra.





## Lecture: Complex Exponentials

## Introduction

- The complex exponential signal is defined as

$$x(t) = A\exp(j(2\pi ft + \phi)).$$

  - As with sinusoids, $A$, $f$, and $\phi$ are (real-valued) amplitude, frequency, and phase.
- By Euler's relationship, it is closely related to sinusoidal signals

$$x(t) = A\cos(2\pi ft + \phi) + jA\sin(2\pi ft + \phi).$$

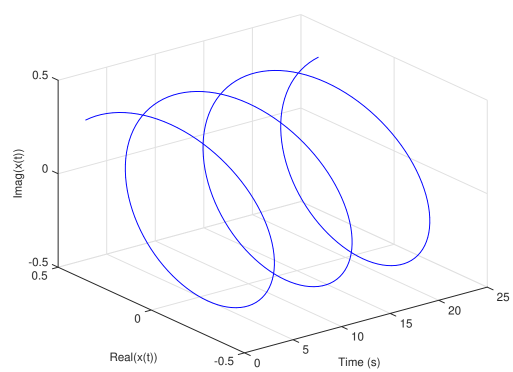
- We will leverage the benefits the complex representation provides over sinusoids:
  - Avoid trigonometry,
  - Replace with simple algebra,
  - Visualization in the complex plane.





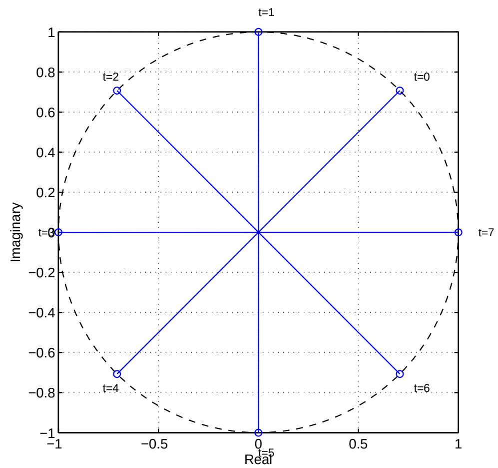
## Plot of Complex Exponential

$$x(t) = 1 \cdot \exp(j(2\pi/8t + \pi/4))$$




Since $x(t)$ is complex-valued, both real and imaginary parts are functions of time.

## Complex Plane




$$x(t) = 1 \cdot e^{j(2\pi/8t+\pi/4)}$$

We can think of a complex expontial as signals that rotate along a circle in the complex plane.





## Expressing Sinusoids through Complex Exponentials

- There are two ways to write a sinusoidal signal in terms of complex exponentials.
- **Real part:**

$$A\cos(2\pi ft + \phi) = \mathrm{Re}\{A\exp(j(2\pi ft + \phi))\}.$$

- **Inverse Euler:**

$$A\cos(2\pi ft + \phi) = \frac{A}{2}(\exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi)))$$

- Both expressions are useful and will be important throughout the course.

## Phasors

- Phasors are **not** directed-energy weapons first seen in the original Star Trek movie.
  - That would be *phasers*!
- Phasors are the complex amplitudes of complex exponential signals:

$$x(t) = A\exp(j(2\pi ft + \phi)) = Ae^{j\phi}\exp(j2\pi ft).$$

- The phasor of this complex exponential is $X = Ae^{j\phi}$.
- Thus, phasors capture both amplitude $A$ and phase $\phi$ – in polar coordinates.
  - The real and imaginary parts of the phasor $X = Ae^{j\phi}$ are referred to as the *in-phase* (I) and *quadrature* (Q) components of $X$, respectively:

$$X = I + jQ = A\cos(\phi) + jA\sin(\phi)$$





## Phasor Notation for Complex Exponentials

- The complex exponential signal

$$x(t) = A\exp(j(2\pi ft + \phi)) = Ae^{j\phi}\exp(j2\pi ft)$$

is characterized completely by the combination of
  - phasor $X = Ae^{j\phi}$
  - frequency $f$
- We will frequently use this observation to denote a complex exponential by providing the pair of phasor and frequency:

$$(Ae^{j\phi}, f)$$

  - We will refer to this notation as the *spectrum representation* of the complex exponential $x(t)$

## From Sinusoids to Phasors

- A sinusoid can be written as

$$A\cos(2\pi ft+\phi) = \frac{A}{2}(\exp(j(2\pi ft+\phi)) + \exp(-j(2\pi ft+\phi))).$$

- This can be rewritten to provide

$$A\cos(2\pi ft+\phi) = \frac{Ae^{j\phi}}{2}\exp(j2\pi ft) + \frac{Ae^{-j\phi}}{2}\exp(-j2\pi ft).$$

- Thus, a sinusoid is composed of two complex exponentials
  - One with frequency $f$ and phasor $\frac{Ae^{j\phi}}{2}$,
    - rotates counter-clockwise in the complex plane;
  - one with frequency $-f$ and phasor $\frac{Ae^{-j\phi}}{2}$.
    - rotates clockwise in the complex plane;
  - Note that the two phasors are conjugate complexes of each other.





## Exercise

- Write
$$x(t) = 3\cos(2\pi 10t - \pi/3)$$
as a sum of two complex exponentials.
- For each of the two complex exponentials, find the frequency and the phasor.
- Repeat for
$$y(t) = 2\sin(2\pi 10t + \pi/4)$$
- What are the in-phase and quadrature signals of
$$z(t) = 5e^{j\pi/3}\exp(j2\pi 10t)$$

## Answers to Exercise

- $$x(t) = 3\cos(2\pi 10t - \pi/3) = \frac{3}{2}e^{-j\pi/3}e^{j2\pi 10t} + \frac{3}{2}e^{j\pi/3}e^{-j2\pi 10t}$$
  as a sum of two complex exponentials.
- Phasor-frequency pairs: $(\frac{3}{2}e^{-j\pi/3}, 10)$ and $(\frac{3}{2}e^{j\pi/3}, -10)$
- $$y(t) = 2\sin(2\pi 10t + \pi/4) = 2\cos(2\pi 10t - \pi/4) = 1e^{-j\pi/4}e^{j2\pi 10t} + 1e^{j\pi/4}e^{-j2\pi 10t}$$
- $$z(t) = 5e^{j\pi/3}\exp(j2\pi 10t) = (\frac{5}{2} + j\frac{5\sqrt{2}}{2})\exp(j2\pi 10t)$$
  Thus, $I = \frac{5}{2}$ and $Q = \frac{5\sqrt{2}}{2}$





## Lecture: The Phasor Addition Rule

## Problem Statement

- It is often required to add two or more sinusoidal signals.
- When all sinusoids have the same frequency then the problem simplifies.
  - This problem comes up very often, e.g., in AC circuit analysis (ECE 280) and later in the class (chapter 5).
- Starting point: sum of sinusoids

$$x(t) = A_1 \cos(2\pi f t + \phi_1) + \ldots + A_N \cos(2\pi f t + \phi_N)$$

  - Note that all frequencies $f$ are the same (no subscript).
  - Amplitudes $A_i$ phases $\phi_i$ are different in general.
  - Short-hand notation using summation symbol ($\sum$):

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i)$$





## The Phasor Addition Rule

- The phasor addition rule implies that there exist an amplitude $A$ and a phase $\phi$ such that

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) = A \cos(2\pi f t + \phi)$$

- **Interpretation:** The sum of sinusoids of the same frequency but different amplitudes and phases is
  - a single sinusoid of the same frequency.
  - The phasor addition rule specifies how the amplitude $A$ and the phase $\phi$ depends on the original amplitudes $A_i$ and $\phi_i$.
- **Example:** We showed earlier (by means of an unpleasant computation involving trig identities) that:

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2) = 5 \cos(2\pi f t + 53^o)$$

## Prerequisites

- We will need two simple prerequisites before we can derive the phasor addition rule.
  1. Any sinusoid can be written in terms of complex exponentials as follows
     $$A\cos(2\pi ft+\phi) = \text{Re}\{Ae^{j(2\pi ft+\phi)}\} = \text{Re}\{Ae^{j\phi}e^{j2\pi ft}\}.$$
     Recall that $Ae^{j\phi}$ is called a phasor (complex amplitude).
  2. For any complex numbers $X_1, X_2, \ldots, X_N$, the real part of the sum equals the sum of the real parts.
     $$\text{Re}\left\{\sum_{i=1}^{N} X_i\right\} = \sum_{i=1}^{N}\text{Re}\{X_i\}.$$
     - This should be obvious from the way addition is defined for complex numbers.
       $$(x_1+jy_1)+(x_2+jy_2) = (x_1+x_2)+j(y_1+y_2).$$





## Deriving the Phasor Addition Rule

- **Objective**: We seek to establish that
  $$\sum_{i=1}^{N} A_i\cos(2\pi ft+\phi_i) = A\cos(2\pi ft+\phi)$$
  and determine how $A$ and $\phi$ are computed from the $A_i$ and $\phi_i$.

## Deriving the Phasor Addition Rule

- **Step 1:** Using the first pre-requisite, we replace the sinusoids with complex exponentials

$$\begin{aligned}\sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) &= \sum_{i=1}^{N} \text{Re}\{A_i e^{j(2\pi f t + \phi_i)}\} \\ &= \sum_{i=1}^{N} \text{Re}\{A_i e^{j\phi_i} e^{j2\pi f t}\}.\end{aligned}$$



Sinusoidal Signals | Sums of Sinusoids | Complex Exponential Signals

## Deriving the Phasor Addition Rule

- **Step 2:** The second prerequisite states that the sum of the real parts equals the the real part of the sum

$$\sum_{i=1}^{N} \text{Re}\{A_i e^{j\phi_i} e^{j2\pi f t}\} = \text{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi f t}\right\}.$$

## Deriving the Phasor Addition Rule

- **Step 3:** The exponential $e^{j2\pi ft}$ appears in all the terms of the sum and can be factored out

$$\text{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\} = \text{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\}$$

  - The term $\sum_{i=1}^{N} A_i e^{j\phi_i}$ is just the sum of complex numbers in polar form.
  - The sum of complex numbers is just a complex number $X$ which can be expressed in polar form as $X = Ae^{j\phi}$.
  - Hence, amplitude $A$ and phase $\phi$ must satisfy

$$Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$$





## Deriving the Phasor Addition Rule

- Note
  - computing $\sum_{i=1}^{N} A_i e^{j\phi_i}$ requires converting $A_i e^{j\phi_i}$ to rectangular form,
  - the result will be in rectangular form and must be converted to polar form $Ae^{j\phi}$.

## Deriving the Phasor Addition Rule

- **Step 4:** Using $Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$ in our expression for the sum of sinusoids yields:

$$\begin{aligned}\text{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\} &= \text{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\} \\ &= \text{Re}\left\{Ae^{j(2\pi ft+\phi)}\right\} \\ &= A\cos(2\pi ft+\phi).\end{aligned}$$

- Note: the above result shows that the sum of sinusoids of the same frequency is a sinusoid of the same frequency.





## Applying the Phasor Addition Rule

- Applicable only when sinusoids of same frequency need to be added!
- **Problem:** Simplify

$$x(t) = A_1\cos(2\pi ft+\phi_1) + \ldots A_N\cos(2\pi ft+\phi_N)$$

- **Solution:** proceeds in 4 steps
  1. Extract phasors: $X_i = A_i e^{j\phi_i}$ for $i = 1, \ldots, N$.
  2. Convert phasors to rectangular form: $X_i = A_i\cos\phi_i + jA_i\sin\phi_i$ for $i = 1, \ldots, N$.
  3. Compute the sum: $X = \sum_{i=1}^{N} X_i$ by adding real parts and imaginary parts, respectively.
  4. Convert result $X$ to polar form: $X = Ae^{j\phi}$.
- **Conclusion:** With amplitude $A$ and phase $\phi$ determined in the final step

$$x(t) = A\cos(2\pi ft+\phi).$$

## Example

- **Problem:** Simplify

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2)$$

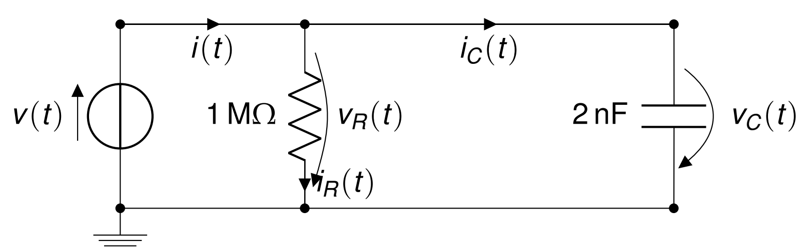
- **Solution:**
  1. Extract Phasors: $X_1 = 3e^{j0} = 3$ and $X_2 = 4e^{j\pi/2}$.
  2. Convert to rectangular form: $X_1 = 3$ $X_2 = 4j$.
  3. Sum: $X = X_1 + X_2 = 3 + 4j$.
  4. Convert to polar form: $A = \sqrt{3^2 + 4^2} = 5$ and $\phi = \arctan(\frac{4}{3}) \approx 53^o$ $(\frac{53}{180}\pi)$.
- **Result:**

$$x(t) = 5\cos(2\pi f t + 53^o).$$



Sinusoidal Signals Sums of Sinusoids Complex Exponential Signals

## The Circuits Example




- For $v(t) = 1\,\text{V} \cdot \cos(2\pi 1\,\text{kHz} \cdot t)$, find the current $i(t)$.

## Problem Formulation with Phasors

- Source:

$$v(t) = 1\,\text{V} \cdot \cos(2\pi 1\,\text{kHz} \cdot t) = \text{Re}\{1\,\text{V} \cdot \exp(j2\pi 1\,\text{kHz} \cdot t)\}$$

  $\Rightarrow$ phasor: $V = 1\,\text{V}e^{j0}$
- Kirchhoff's voltage law: $v(t) = v_R(t) = v_C(t)$;
  $\Rightarrow$ phasors: $V = V_R = V_C$.
- Resistor: $i_R(t) = \frac{v_R(t)}{R}$;
  $\Rightarrow$ phasor: $I_R = \frac{V_R}{R}$
- Capacitor: $i_C(t) = C\frac{dv_C(t)}{dt}$;
  $\Rightarrow$ phasor: $I_C = C \cdot V \cdot j2\pi \cdot 1\,\text{kHz}$
  - Because $\frac{d \exp(j2\pi 1\,\text{kHz} \cdot t)}{dt} = j2\pi 1\,\text{kHz} \cdot \exp(j2\pi 1\,\text{kHz} \cdot t)$
- Kirchhoff's current law: $i(t) = i_R(t) + i_C(t)$;
  $\Rightarrow$ phasors: $I = I_R + I_C$.





## Problem Formulation with Phasors

- Therefore,

$$\begin{aligned} I &= \frac{V}{R} + C \cdot V \cdot j2\pi \cdot 1\,\text{kHz} \\ &= \frac{1\,\text{V}}{1\,\text{M}\Omega} + j2\pi \cdot 1\,\text{kHz} \cdot 2\,\text{nF} \cdot 1\,\text{V} \\ &= 1\,\mu\text{A} + j4\pi\,\mu\text{A} \end{aligned}$$

- Convert to polar form:

$$1\,\mu\text{A} + j4\pi\,\mu\text{A} = 12.6\,\mu\text{A} \cdot e^{j0.47\pi}$$

  Using:
  - $\sqrt{1^2 + (4\pi)^2} \approx 12.6$
  - $\tan^{-1}((4\pi)) \approx 0.47\pi$
- Thus, $i(t) \approx 12.6\,\mu\text{A} \cos(2\pi 1\,\text{kHz} \cdot t + 0.47 \cdot \pi)$.

## Exercise

- Simplify

$$x(t) = 10\cos(20\pi t + \frac{\pi}{4}) + 10\cos(20\pi t + \frac{3\pi}{4}) + 20\cos(20\pi t - \frac{3\pi}{4}).$$

- Answer:

$$x(t) = 10\sqrt{2}\cos(20\pi t + \pi).$$





# Part III

## Spectrum Representation of Signals

## Lecture: Sums of Sinusoids (of different frequency)





## Introduction

- To this point we have focused on sinusoids of identical frequency $f$

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i).$$

  - Note that the frequency $f$ does not have a subscript $i$!
- Showed (via phasor addition rule) that the above sum can always be written as a single sinusoid of frequency $f$.

## Introduction

- We will consider sums of sinusoids of different frequencies:

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

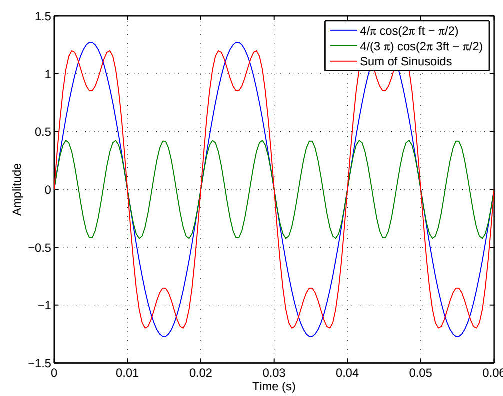
  - Note the subscript on the frequencies $f_i$!
  - This apparently minor difference has dramatic consequences.





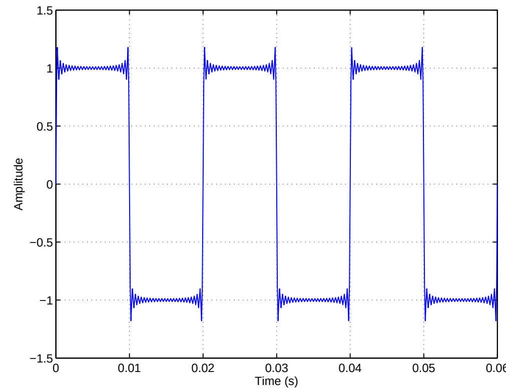
## Sum of Two Sinusoids

$$x(t) = \frac{4}{\pi}\cos(2\pi ft - \pi/2) + \frac{4}{3\pi}\cos(2\pi 3ft - \pi/2)$$



## Sum of 25 Sinusoids

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$








## Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusods in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids
    - of frequencies $(2n-1) \cdot f$, and
    - amplitudes $\frac{4}{(2n-1)\pi}$.
    - (This is not obvious $\rightarrow$ Fourier Series).

## Non-sinusoidal Signals as Sums of Sinusoids

- The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.
- **Basic idea:** list the *"ingredients"* of a signal by specifying
  - amplitudes and phases, as well as
  - frequencies of the sinusoids in the sum.





## The Spectrum of a Sum of Sinusoids

- Begin with the sum of sinusoids introduced earlier

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

  where we have broken out a possible constant term.
  - The term $A_0$ can be thought of as corresponding to a sinusoid of frequency zero.
- Using the *inverse Euler formula*, we can replace the sinusoids by complex exponentials

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

  where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

## The Spectrum of a Sum of Sinusoids (cont'd)

- Starting with

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

- The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

$$X(f) = \{(X_0, 0), (\frac{X_1}{2}, f_1), (\frac{X_1^*}{2}, -f_1), \ldots, (\frac{X_N}{2}, f_N), (\frac{X_N^*}{2}, -f_N)\}$$





## Example

- Consider the signal

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

- Using the inverse Euler relationship

$$\begin{aligned} x(t) = 3 \quad &+ \quad \tfrac{5}{2} e^{-j\pi/2} \exp(j2\pi 10t) \quad + \quad \tfrac{5}{2} e^{j\pi/2} \exp(-j2\pi 10t) \\ &+ \quad \tfrac{7}{2} e^{j\pi/4} \exp(j2\pi 25t) \quad + \quad \tfrac{7}{2} e^{-j\pi/4} \exp(-j2\pi 25t). \end{aligned}$$

- Hence,

$$\begin{aligned} X(f) = \{(3, 0), \quad & (\tfrac{5}{2} e^{-j\pi/2}, 10), (\tfrac{5}{2} e^{j\pi/2}, -10), \\ & (\tfrac{7}{2} e^{j\pi/4}, 25), (\tfrac{7}{2} e^{-j\pi/4}, -25)\} \end{aligned}$$

## Exercise

- Find the spectrum of the signal:

$$x(t) = 6 + 4\cos(10\pi t + \pi/3) + 5\cos(20\pi t - \pi/7).$$





## Time-domain and Frequency-domain

- Signals are *naturally* observed in the time-domain.
- A signal can be illustrated in the time-domain by plotting it as a function of time.
- The frequency-domain provides an alternative perspective of the signal based on sinusoids:
  - Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
  - The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
  - Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of theses parameters.
    - Actually, we list pairs of complex amplitudes ($Ae^{j\phi}$) and frequencies $f$ and refer to this list as $X(f)$.

# Time-domain and Frequency-domain

- It is possible (but not necessarily easy) to find $X(f)$ from $x(t)$: this is called Fourier or spectrum analysis.
- Similarly, one can construct $x(t)$ from the spectrum $X(f)$: this is called Fourier synthesis.
- Notation: $x(t) \leftrightarrow X(f)$.
- Example (from earlier):
  - **Time-domain:** signal

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

  - **Frequency Domain:** spectrum

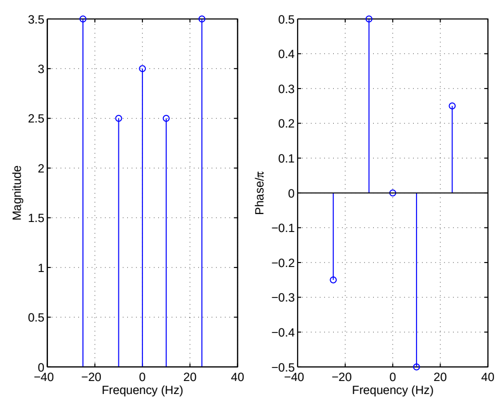
$$X(f) = \{(3, 0), \quad (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}$$





# Plotting a Spectrum

- To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.
- Sometimes the phase is plotted versus frequency as well.



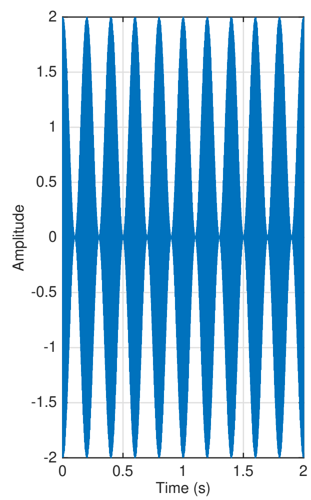
## Why Bother with the Frequency-Domain?

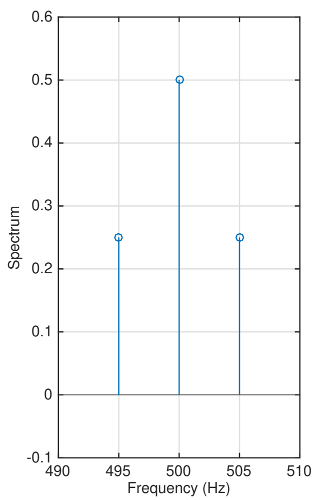
- In many applications, the frequency contents of a signal is very important.
  - For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
  - Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.
- Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).
- We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.
  - Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.



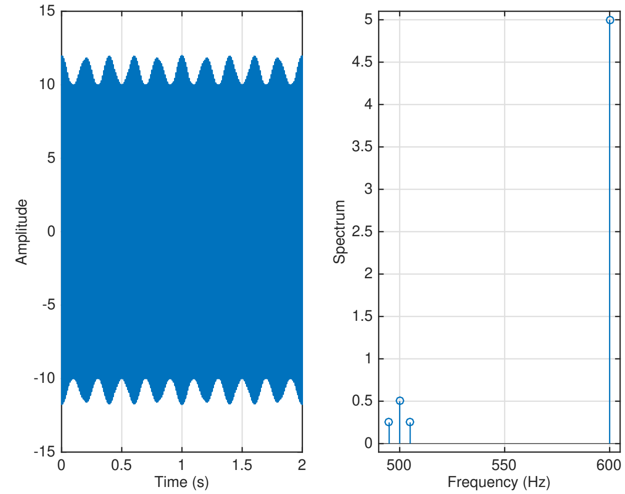
Sum of Sinusoidal Signals | Time and Frequency-Domain | Periodic Signals | Time-Frequency Spectrum | Operations on Spectrum

## Example: Original signal



## Example: Corrupted signal








## Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- *Ingredients* are given by the spectrum

$$X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\}$$

  Each pair indicates one complex exponential component by listing its frequency and complex amplitude.
- *Instructions* for combining the ingredients and producing the (time-domain) signal:

$$x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t).$$

- Always simplify the expression you obtain!

## Example

- Problem: Find the signal $x(t)$ corresponding to

$$X(f) = \{(3,0), \quad (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}$$

- Solution:

$$x(t) = 3 \quad + \tfrac{5}{2}e^{-j\pi/2}e^{j2\pi 10t} + \tfrac{5}{2}e^{j\pi/2}e^{-j2\pi 10t} + \tfrac{7}{2}e^{j\pi/4}e^{j2\pi 25t} + \tfrac{7}{2}e^{-j\pi/4}e^{-j2\pi 25t}$$

- Which simplifies to:

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$





## Exercise

- Find the signal with the spectrum:

$$X(f) = \{(5,0), \quad (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), (\tfrac{5}{2}e^{j\pi/4}, 15), (\tfrac{5}{2}e^{-j\pi/4}, -15)$$

## Analysis: From Time to Frequency-Domain

- The objective of spectrum or Fourier analysis is to find the spectrum of a time-domain signal.
- We will restrict ourselves to signals $x(t)$ that are sums of sinusoids

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- We have already shown that such signals have spectrum:

$$X(f) = \{(X_0, 0), (\frac{1}{2}X_1, f_1), (\frac{1}{2}X_1^*, -f_1), \ldots, (\frac{1}{2}X_N, f_N), (\frac{1}{2}X_N^*, -f_N)$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

- We will investigate some interesting signals that can be written as a sum of sinusoids.



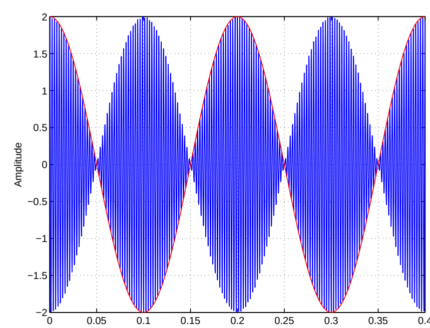


## Beat Notes

- Consider the signal

$$x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t).$$

- This signal does not have the form of a sum of sinusoids; hence, we can not determine it's spectrum immediately.



## MATLAB Code for Beat Notes

```
% Parameters
fs  = 8192;
dur = 2;

f1 = 5;
f2 = 400;
A  = 2;

NP  = round(2*fs/f1); % number of samples to plot

% time axis and signal
tt=0:1/fs:dur;
xx = A*cos(2*pi*f1*tt).*cos(2*pi*f2*tt);

plot(tt(1:NP),xx(1:NP),tt(1:NP),A*cos(2*pi*f1*tt(1:NP)),'r')
xlabel('Time(s)')
ylabel('Amplitude')
grid
```





## Beat Notes as a Sum of Sinusoids

- Using the inverse Euler relationships, we can write

$$\begin{aligned} x(t) &= 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t) \\ &= 2 \cdot \tfrac{1}{2} \cdot (e^{j2\pi 5t} + e^{-j2\pi 5t}) \cdot \tfrac{1}{2} \cdot (e^{j2\pi 400t} + e^{-j2\pi 400t}). \end{aligned}$$

- Multiplying out yields:

$$x(t) = \frac{1}{2}(e^{j2\pi 405t} + e^{-j2\pi 405t}) + \frac{1}{2}(e^{j2\pi 395t} + e^{-j2\pi 395t}).$$

- Applying Euler's relationship, lets us write:

$$x(t) = \cos(2\pi 405t) + \cos(2\pi 395t).$$

## Spectrum of Beat Notes

- We were able to rewrite the beat notes as a sum of sinusoids

$$x(t) = \cos(2\pi 405 t) + \cos(2\pi 395 t).$$

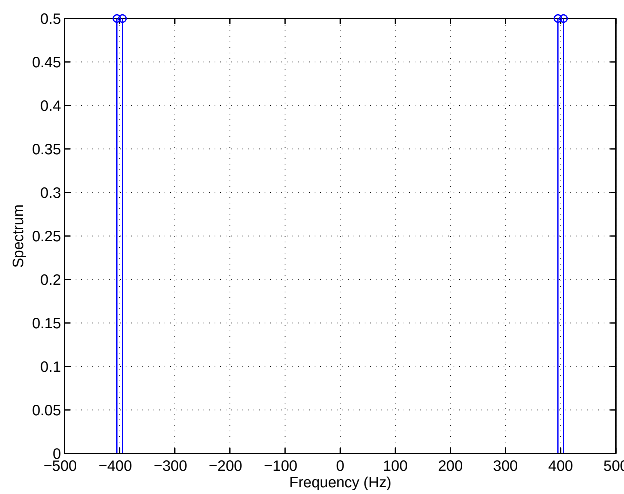
- Note that the frequencies in the sum, 395 Hz and 405 Hz, are the sum and difference of the frequencies in the original product, 5 Hz and 400 Hz.
- It is now straightforward to determine the spectrum of the beat notes signal:

$$X(f) = \{(\frac{1}{2}, 405), (\frac{1}{2}, -405), (\frac{1}{2}, 395), (\frac{1}{2}, -395)\}$$





## Spectrum of Beat Notes



## Amplitude Modulation

- Amplitude Modulation is used in communication systems.
- The objective of amplitude modulation is to move the spectrum of a signal $m(t)$ from low frequencies to high frequencies.
  - The message signal $m(t)$ may be a piece of music; its spectrum occupies frequencies below 20 KHz.
  - For transmission by an AM radio station this spectrum must be moved to approximately 1 MHz.





## Amplitude Modulation

- Conventional amplitude modulation proceeds in two steps:
  1. A constant $A$ is added to $m(t)$ such that $A + m(t) > 0$ for all $t$.
  2. The sum signal $A + m(t)$ is multiplied by a sinusoid $\cos(2\pi f_c t)$, where $f_c$ is the radio frequency assigned to the station.
- Consequently, the transmitted signal has the form:

$$x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

## Amplitude Modulation

- We are interested in the spectrum of the AM signal.
- However, we cannot compute $X(f)$ for arbitrary message signals $m(t)$.
- For the special case $m(t) = \cos(2\pi f_m t)$ we can find the spectrum.
  - To mimic the radio case, $f_m$ would be a frequency in the audible range.
- As before, we will first need to express the AM signal $x(t)$ as a sum of sinusoids.





## Amplitude Modulated Signal

- For $m(t) = \cos(2\pi f_m t)$, the AM signal equals

$$x(t) = (A + \cos(2\pi f_m t)) \cdot \cos(2\pi f_c t).$$

- This simplifies to

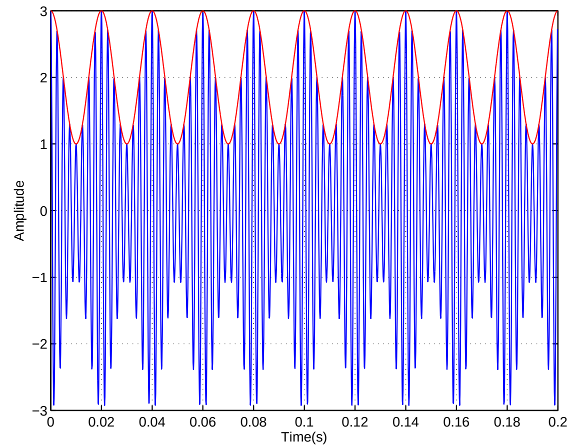
$$x(t) = A \cdot \cos(2\pi f_c t) + \cos(2\pi f_m t) \cdot \cos(2\pi f_c t).$$

- Note that the second term of the sum is a beat notes signal with frequencies $f_m$ and $f_c$.
- We know that beat notes can be written as a sum of sinusoids with frequencies equal to the sum and difference of $f_m$ and $f_c$:

$$x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2}\cos(2\pi(f_c + f_m)t) + \frac{1}{2}\cos(2\pi(f_c - f_m)t).$$

## Plot of Amplitude Modulated Signal

For $A = 2$, $fm = 50$, and $fc = 400$, the AM signal is plotted below.








## Spectrum of Amplitude Modulated Signal

- The AM signal is given by

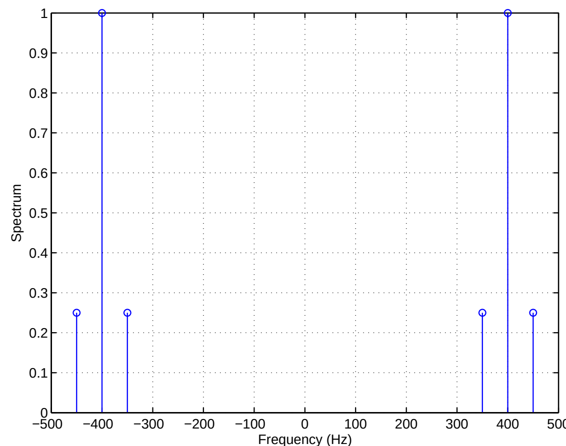
$$x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2}\cos(2\pi(f_c + f_m)t) + \frac{1}{2}\cos(2\pi(f_c - f_m)t).$$

- Thus, its spectrum is

$$X(f) = \{ \quad (\tfrac{A}{2}, f_c), (\tfrac{A}{2}, -f_c), (\tfrac{1}{4}, f_c + f_m), (\tfrac{1}{4}, -f_c - f_m), (\tfrac{1}{4}, f_c - f_m), (\tfrac{1}{4}, -f_c + f_m)\}$$

## Spectrum of Amplitude Modulated Signal

For $A = 2$, $fm = 50$, and $fc = 400$, the spectrum of the AM signal is plotted below.








## Spectrum of Amplitude Modulated Signal

- It is interesting to compare the spectrum of the signal before modulation and after multiplication with $\cos(2\pi f_c t)$.
- The signal $s(t) = A + m(t)$ has spectrum

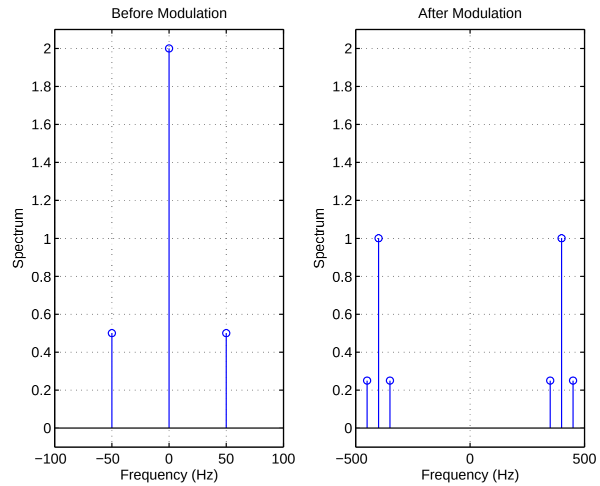
$$S(f) = \{(A, 0), (\frac{1}{2}, 50), (\frac{1}{2}, -50)\}.$$

- The modulated signal $x(t)$ has spectrum

$$X(f) = \{ \quad (\tfrac{A}{2}, 400), (\tfrac{A}{2}, -400), \\ (\tfrac{1}{4}, 450), (\tfrac{1}{4}, -450), (\tfrac{1}{4}, 350), (\tfrac{1}{4}, -350)\}$$

- Both are plotted on the next page.

## Spectrum before and after AM








## Spectrum before and after AM

- Comparison of the two spectra shows that amplitude modulation indeed moves a spectrum from low frequencies to high frequencies.
- Note that the shape of the spectrum is precisely preserved.
- Amplitude modulation can be described concisely by stating:
  - Half of the original spectrum is shifted by $f_c$ to the right, and the other half is shifted by $f_c$ to the left.
- **Question:** How can you get the original signal back so that you can listen to it.
  - This is called demodulation.

# Lecture: Periodic Signals





## What are Periodic Signals?

- A signal $x(t)$ is called periodic if there is a constant $T_0$ such that
$$x(t) = x(t + T_0) \text{ for all } t.$$
- In other words, a periodic signal repeats itself every $T_0$ seconds.
- The interval $T_0$ is called the fundamental period of the signal.
- The inverse of $T_0$ is the fundamental frequency of the signal.
- Example:
  - A sinusoidal signal of frequency $f$ is periodic with period $T_0 = 1/f$.

## Harmonic Frequencies

- Consider a sum of sinusoids:

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- A special case arises when we constrain all frequencies $f_i$ to be integer multiples of some frequency $f_0$:

$$f_i = i \cdot f_0.$$

- The frequencies $f_i$ are then called harmonic frequencies of $f_0$.
- We will show that sums of sinusoids with frequencies that are harmonics are periodic.




Sum of Sinusoidal Signals | Time and Frequency-Domain | Periodic Signals | Time-Frequency Spectrum | Operations on Spectrum

## Harmonic Signals are Periodic

- To establish periodicity, we must show that there is $T_0$ such $x(t) = x(t + T_0)$.
- Begin with

$$\begin{aligned} x(t+T_0) &= A_0 + \textstyle\sum_{i=1}^{N} A_i \cos(2\pi f_i (t+T_0) + \phi_i) \\ &= A_0 + \textstyle\sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i) \end{aligned}$$

- Now, let $f_0 = 1/T_0$ and use the fact that frequencies are harmonics: $f_i = i \cdot f_0$.

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## Harmonic Signals are Periodic

- Then, $f_i \cdot T_0 = i \cdot f_0 \cdot T_0 = i$ and hence

$$\begin{aligned} x(t+T_0) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i) \\ &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi i + \phi_i) \end{aligned}$$

- We can drop the $2\pi i$ terms and conclude that $x(t+T_0) = x(t)$.
- **Conclusion:** A signal of the form

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i \cdot f_0 t + \phi_i)$$

is periodic with period $T_0 = 1/f_0$.



Sum of Sinusoidal Signals Time and Frequency-Domain Periodic Signals Time-Frequency Spectrum Operations on Spectrum

## Finding the Fundamental Frequency

- Often one is given a set of frequencies $f_1, f_2, \ldots, f_N$ and is required to find the fundamental frequency $f_0$.
- Specifically, this means one must find a frequency $f_0$ and integers $n_1, n_2, \ldots, n_N$ such that all of the following equations are met:

$$\begin{aligned} f_1 &= n_1 \cdot f_0 \\ f_2 &= n_2 \cdot f_0 \\ &\vdots \\ f_N &= n_N \cdot f_0 \end{aligned}$$

- Note that there isn't always a solution to the above problem.
  - However, if all frequencies are integers a solution exists.
  - Even if all frequencies are rational a solution exists.

## Example

- Find the fundamental frequency for the set of frequencies $f_1 = 12$, $f_2 = 27$, $f_3 = 51$.
- Set up the equations:

$$\begin{aligned} 12 &= n_1 \cdot f_0 \\ 27 &= n_2 \cdot f_0 \\ 51 &= n_3 \cdot f_0 \end{aligned}$$

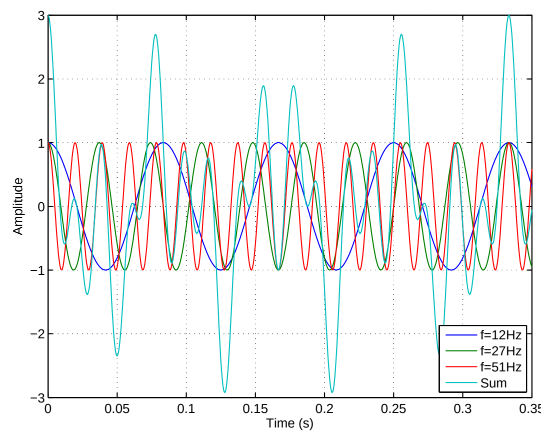
- Try the solution $n_1 = 1$; this would imply $f_0 = 12$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 2$; this would imply $f_0 = 6$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 3$; this would imply $f_0 = 4$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 4$; this would imply $f_0 = 3$. This **can** satisfy the other two equations with $n_2 = 9$ and $n_3 = 17$.





## Example

- Note that the three sinusoids complete a cycle at the same time at $T_0 = 1/f_0 = 1/3s$.



## A Few Things to Note

- Note that the fundamental frequency $f_0$ that we determined is the greatest common divisor (gcd) of the original frequencies.
  - $f_0 = 3$ is the gcd of $f_1 = 12$, $f_2 = 27$, and $f_3 = 51$.
- The integers $n_i$ are the number of full periods (cycles) the sinusoid of freqency $f_i$ completes in the fundamental period $T_0 = 1/f_0$.
  - For example, $n_1 = f_1 \cdot T_0 = f_1 \cdot 1/f_0 = 4$.
  - The sinusoid of frequency $f_1$ completes $n_1 = 4$ cycles during the period $T_0$.





## Exercise

- Find the fundamental frequency for the set of frequencies $f_1 = 2$, $f_2 = 3.5$, $f_3 = 5$.

## Fourier Series

- We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.
- One can turn this statement around and arrive at a very important result:

  *Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.*

- The resulting sum is called the Fourier Series of the signal.
- Put differently, a periodic signal can always be written in the form

$$\begin{aligned} x(t) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i f_0 t + \phi_i) \\ &= X_0 + \sum_{i=1}^{N} X_i e^{j2\pi i f_0 t} + X_i^* e^{-j2\pi i f_0 t} \end{aligned}$$

  with $X_0 = A_0$ and $X_i = \frac{A_i}{2} e^{j\phi_i}$.



Sum of Sinusoidal Signals | Time and Frequency-Domain | Periodic Signals | Time-Frequency Spectrum | Operations on Spectrum

## Fourier Series

- For a periodic signal the complex amplitudes $X_i$ can be computed using a (relatively) simple formula.
- Specifically, for a periodic signal $x(t)$ with fundamental period $T_0$ the complex amplitudes $X_i$ are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi i t/T_0} dt.$$

  - Note that the integral above can be evaluated over any interval of length $T_0$.

## Example: Square Wave

- A square wave signal is periodic and between $t = 0$ and $t = T_0$ it equals

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

- From the Fourier Series expansion it follows that $x(t)$ can be written as

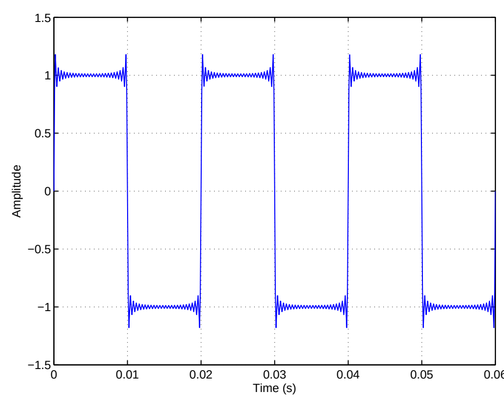
$$x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$





## 25-Term Approximation to Square Wave

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$



## Limitations of Sum-of-Sinusoid Signals

- So far, we have considered only signals that can be written as a sum of sinusoids.

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- For such signals, we are able to compute the spectrum.
- Note, that signals of this form
  - are assumed to last forever, i.e., for $-\infty < t < \infty$,
  - and their spectrum never changes.
- While such signals are important and useful conceptually, they don't describe real-world signals accurately.
- Real-world signals
  - are of finite duration,
  - their spectrum changes over time.





## Musical Notation

- Musical notation ("sheet music") provides a way to represent real-world signals: a piece of music.
- As you know, sheet music
  - places notes on a scale to reflect the *frequency* of the tone to be played,
  - uses differently shaped note symbols to indicate the *duration* of each tone,
  - provides the order in which notes are to be played.
- In summary, musical notation captures how the spectrum of the music-signal changes over time.
- We cannot write signals whose spectrum changes with time as a sum of sinusoids.
  - A *static* spectrum is insufficient to describe such signals.
- Alternative: time-frequency spectrum

## Example: Musical Scale

| Note | C | D | E | F | G | A | B | C |
|---|---|---|---|---|---|---|---|---|
| Frequency (Hz) | 262 | 294 | 330 | 349 | 392 | 440 | 494 | 523 |

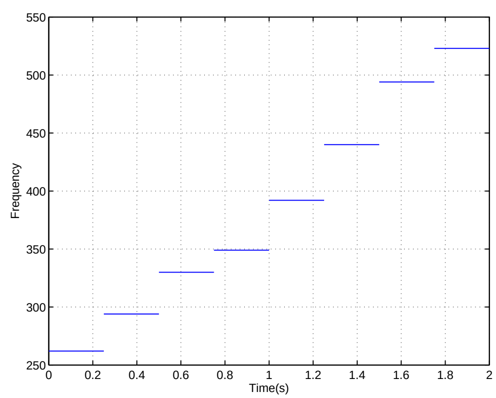
Table: Musical Notes and their Frequencies





## Example: Musical Scale

- If we play each of the notes for 250 ms, then the resulting signal can be summarized in the time-frequency spectrum below.



## MATLAB Spectrogram Function

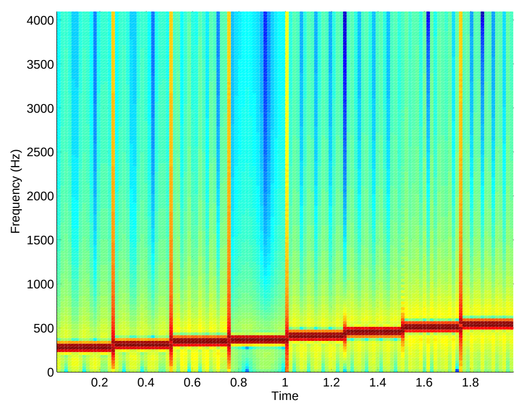
- MATLAB has a function `spectrogram` that can be used to compute the time-frequency spectrum for a given signal.
  - The resulting plots are similar to the one for the musical scale on the previous slide.
- Typically, you invoke this function as
  `spectrogram( xx, 256, 128, 256, fs,'yaxis'),`
  where `xx` is the signal to be analyzed and `fs` is the sampling frequency.
- The spectrogram for the musical scale is shown on the next slide.





## Spectrogram: Musical Scale

- The color indicates the magnitude of the spectrum at a given time and frequency.



## Chirp Signals

- **Objective:** construct a signal such that its frequency increases with time.
- **Starting Point:** A sinusoidal signal has the form:

$$x(t) = A\cos(2\pi f_0 t + \phi).$$

- We can consider the argument of the cos as a time-varying phase function

$$\Psi(t) = 2\pi f_0 t + \phi.$$

- **Question:** What happens when we allow more general functions for $\Psi(t)$?
  - For example, let

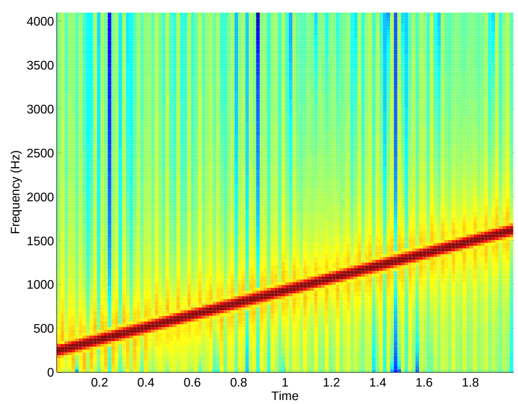
$$\Psi(t) = 700\pi t^2 + 440\pi t + \phi.$$





## Spectrogram: $\cos(\Psi(t))$

- **Question:** How is he time-frequency spectrum related to $\Psi(t)$?



## Instantaneous Frequency

- For a regular sinusoid, $\Psi(t) = 2\pi f_0 t + \phi$ and the frequency equals $f_0$.
- This suggests as a possible relationship between $\Psi(t)$ and $f_0$

$$f_0 = \frac{1}{2\pi}\frac{d}{dt}\Psi(t).$$

- If the above derivative is not a constant, it is called the instantaneous frequency of the signal, $f_i(t)$.
- **Example:** For $\Psi(t) = 700\pi t^2 + 440\pi t + \phi$ we find

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}(700\pi t^2 + 440\pi t + \phi) = 700t + 220.$$

  - This describes precisely the red line in the spectrogram on the previous slide.





## Constructing a Linear Chirp

- **Objective:** Construct a signal such that its frequency is initially $f_1$ and increases linear to $f_2$ after $T$ seconds.
- **Solution:** The above suggests that

$$f_i(t) = \frac{f_2 - f_1}{T} t + f_1.$$

- Consequently, the phase function $\Psi(t)$ must be

$$\Psi(t) = 2\pi \frac{f_2 - f_1}{2T} t^2 + 2\pi f_1 t + \phi$$

  - Note that $\phi$ has no influence on the spectrum; it is usually set to 0.

## Constructing a Linear Chirp

- **Example:** Construct a linear chirp such that the frequency decreases from 1000 Hz to 200 Hz in 2 seconds.
- The desired signal must be

$$x(t) = \cos(-2\pi 200t^2 + 2\pi 1000t).$$





## Exercise

- Construct a linear chirp such that the frequency increases from 50 Hz to 200 Hz in 3 seconds.
- Sketch the time-frequency spectrum of the following signal

$$x(t) = \cos(2\pi 500t + 100\cos(2\pi 2t))$$

## Signal Operations in the Frequency Domain

- Signal processing implies that we apply *operations* to signals; Examples include:
  - Adding two signals
  - Delaying a signal
  - Multiplying a signal with a complex exponential signal
- **Question:** What does each of these operation do the spectrum of the signal?
  - We will answer that question for some common signal processing operations.





## Scaling a Signal

- Let $x(t)$ be a signal with spectrum $X(f) = \{(X_n, f_n)\}_n$.
- **Question:** If $c$ is a scalar constant, what is the spectrum of the signal $y(t) = c \cdot x(t)$?
- Since

$$x(t) = \sum_n X_n \cdot e^{j2\pi f_n t}$$

$$y(t) = c \cdot x(t) = \sum_n c \cdot X_n \cdot e^{j2\pi f_n t}.$$

- Therefore,

$$Y(f) = \{(c \cdot X_n, f_n)\}_n.$$

  - We use the short-hand $Y(f) = c \cdot X(f)$ to denote $\{(c \cdot X_n, f_n)\}_n$.

## Adding Two Signals

- Let $x(t)$ and $y(t)$ be signals with spectra $X(f)$ and $Y(f)$.
- **Question:** What is the spectrum of the signal $z(t) = x(t) + y(t)$?
- Since

$$z(t) = x(t) + y(t) = \sum_n X_n \cdot e^{j2\pi f_n t} + \sum_n Y_n \cdot e^{j2\pi f_n t}$$

$$Z(f) = \{(X_n + Y_n, f_n)\}_n.$$

  - We use the short-hand $Z(f) = X(f) + Y(f)$ to denote $\{(X_n + Y_n, f_n)\}$.
- **Example:** What is the spectrum $Z(f)$ when signals with spectra $X(f) = \{(3,0), (1,1), (1,-1), (2,2), (2,-2)\}$ and $Y(f) = \{(j,1), (-j,-1), (1,3), (1,-3)\}$ are added?





## Delaying a Signal

- Let $x(t)$ be a signal and $X(f) = \{(X_n, f_n)\}_n$ denotes its spectrum.
- **Question:** What is the spectrum of the signal $y(t) = x(t - \tau)$?
- Since

$$y(t) = x(t-\tau) = \sum_n X_n \cdot e^{j2\pi f_n (t-\tau)} = \sum_n X_n e^{-j2\pi f_n \tau} \cdot e^{j2\pi f_n t}$$

  it follows that

$$Y(f) = \{(X_n e^{-j2\pi f_n \tau}, f_n)\}_n.$$

  - Notice that delaying a signal induces *phase shifts* in the spectrum
  - The phase shifts are proportional to the delay $\tau$ and the frequencies $f_n$.

## Delaying a Signal – Example

- **Example:** What is the spectrum $Y(f)$ when the signal with spectrum $X(f) = \{(3,0),(1,1),(1,-1),(2,2),(2,-2)\}$ is shifted by $\tau = \frac{1}{4}$?
- **Answer:**

$$Y(f) = \{(3,0),(-j,1),(j,-1),(-2,2),(-2,-2)\}$$





## Multiplying by a Complex Exponential

- Let $x(t)$ be a signal and $X(f) = \{(c \cdot X_n, f_n)\}_n$ denotes its spectrum.
- **Question:** What is the spectrum of the signal $y(t) = x(t) \cdot e^{j2\pi f_c t}$?
- Since

$$y(t) = x(t) \cdot e^{j2\pi f_c t} = \sum_n X_n \cdot e^{j2\pi f_n t} \cdot e^{j2\pi f_c t} = \sum_n X_n \cdot e^{j2\pi (f_n + f_c) t}$$

it follows that

$$Y(f) = \{X_n, f_n + f_c\}$$

  - Notice that the entire spectrum is shifted by $f_c$, i.e., $Y(f) = X(f + f_c)$.
  - Notice the "symmetry" with the time delay operation — this is called duality.

## Exercise: Spectrum of AM Signal

- We discussed that amplitude modulation *processess* a message signal to produce the transmitted signal $s(t)$:

$$s(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

- Assume that the spectrum of $m(t)$ is $M(f)$.
- **Question:** Use the Spectrum Operations we discussed to express the spectrum $S(f)$ in terms of $M(f)$.
- **Answer:**

$$S(f) = \frac{1}{2}M(f+f_c) + \frac{1}{2}M(f-f_c) + \{(\frac{A}{2}, f_c) + \{(\frac{A}{2}, -f_c)\}$$





# Part IV

## Sampling of Signals

## Lecture: Introduction to Sampling





## Sampling and Discrete-Time Signals

- MATLAB, and other digital processing systems, can not process continuous-time signals.
- Instead, MATLAB requires the continuous-time signal to be converted into a discrete-time signal.
- The conversion process is called sampling.
- To sample a continuous-time signal, we evaluate it at a discrete set of times $t_n = nT_s$, where
  - $n$ is a integer,
  - $T_s$ is called the sampling period (time between samples),
  - $f_s = 1/T_s$ is the sampling rate (samples per second).

## Sampling and Discrete-Time Signals

- Sampling results in a sequence of samples

$$x(nT_s) = A \cdot \cos(2\pi f n T_s + \phi).$$

- Note that the independent variable is now $n$, not $t$.
- To emphasize that this is a discrete-time signal, we write

$$x[n] = A \cdot \cos(2\pi f n T_s + \phi).$$

- Sampling is a straightforward operation.
- We will see that the sampling rate $f_s$ must be chosen with care!





## Sampled Signals in MATLAB

- Note that we have worked with sampled signals whenever we have used MATLAB.
- For example, we use the following MATLAB fragment to generate a sinusoidal signal:

```
fs = 100;
tt = 0:1/fs:3;
xx = 5*cos(2*pi*2*tt + pi/4);
```

- The resulting signal `xx` is a discrete-time signal:
  - The vector `xx` contains the samples, and
  - the vector `tt` specifies the sampling instances: $0, 1/f_s, 2/f_s, \ldots, 3$.
- We will now turn our attention to the impact of the sampling rate $f_s$.

## Example: Three Sinuoids

- **Objective:** In MATLAB, compute sampled versions of three sinusoids:
  1. $x(t) = \cos(2\pi t + \pi/4)$
  2. $x(t) = \cos(2\pi 9t - \pi/4)$
  3. $x(t) = \cos(2\pi 11t + \pi/4)$
- The sampling rate for all three signals is $f_s = 10$.

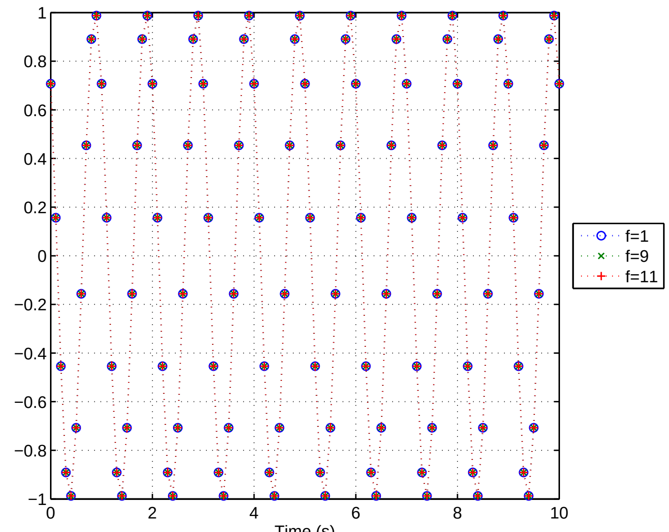




## MATLAB code

```
% plot_SamplingDemo - Sample three sinusoidal signals to
%                     demonstrate the impact of sampling

%% set parameters
fs  = 10;
dur = 10;

%% generate signals
tt  = 0:1/fs:dur;
xx1 = cos(2*pi*tt+pi/4);
xx2 = cos(2*pi*9*tt-pi/4);
xx3 = cos(2*pi*11*tt+pi/4);

%% plot
plot(tt,xx1,':o',tt,xx2,':x',tt,xx3,':+');
xlabel('Time (s)')
grid
legend('f=1','f=9','f=11','Location','EastOutside')
```

## Resulting Plot








## What happened?

- The samples for all three signals are identical: how is that possible?
- Is there a “bug” in the MATLAB code?
  - No, the code is correct.
- **Suspicion:** The problem is related to our choice of sampling rate.
  - To test this suspicion, repeat the experiment with a different sampling rate.
  - We also reduce the duration to keep the number of samples constant - that keeps the plots reasonable.

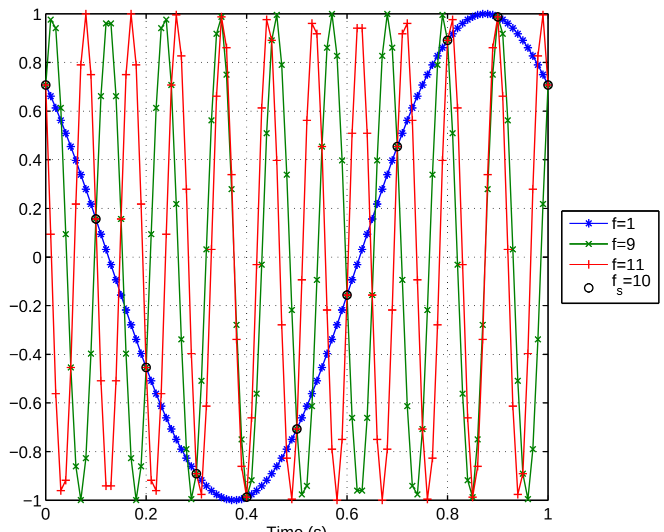
# MATLAB code

```
% plot_SamplingDemoHigh - Sample three sinusoidal signals to
%                         demonstrate the impact of sampling

%% set parameters
fs  = 100;
dur = 1;

%% generate signals
tt  = 0:1/fs:dur;
xx1 = cos(2*pi*tt+pi/4);
xx2 = cos(2*pi*9*tt-pi/4);
xx3 = cos(2*pi*11*tt+pi/4);

%% plots
plot(tt,xx1,'-*',tt,xx2,'-x',tt,xx3,'-+',...
    tt(1:10:end), xx1(1:10:end),'ok');
grid
xlabel('Time␣(s)')
legend('f=1','f=9','f=11','f_s=10','Location','EastOutside')
```





# Resulting Plot



## The Influence of the Sampling Rate

- Now the three sinusoids are clearly distinguishable and lead to different samples.
- Since the only parameter we changed is the sampling rate $f_s$, it must be responsible for the ambiguity in the first plot.
- Notice also that every 10-th sample (marked with a black circle) is identical for all three sinusoids.
  - Since the sampling rate was 10 times higher for the second plot, this explains the first plot.
- It is useful to investigate the effect of sampling mathematically, to understand better what impact it has.
  - To do so, we focus on sampling sinusoidal signals.





## Sampling a Sinusoidal Signal

- A continuous-time sinusoid is given by

$$x(t) = A\cos(2\pi ft + \phi).$$

- When this signal is sampled at rate $f_s$, we obtain the discrete-time signal

$$x[n] = A\cos(2\pi fn/f_s + \phi).$$

- It is useful to define the normalized frequency $\hat{f}_d = \frac{f}{f_s}$, so that

$$x[n] = A\cos(2\pi\hat{f}_d n + \phi).$$

## Three Cases

- We will distinguish between three cases:
  1. $0 \leq \hat{f}_d \leq 1/2$ (Oversampling, this is what we want!)
  2. $1/2 < \hat{f}_d \leq 1$ (Undersampling, folding)
  3. $1 < \hat{f}_d \leq 3/2$ (Undersampling, aliasing)
- This captures the three situations addressed by the first example:
  1. $f = 1, f_s = 10 \Rightarrow \hat{f}_d = 1/10$
  2. $f = 9, f_s = 10 \Rightarrow \hat{f}_d = 9/10$
  3. $f = 11, f_s = 10 \Rightarrow \hat{f}_d = 11/10$
- We will see that all three cases lead to identical samples.





## Oversampling

- When the sampling rate is such that $0 \leq \hat{f}_d \leq 1/2$, then the samples of the sinusoidal signal are given by

$$x[n] = A\cos(2\pi\hat{f}_d n + \phi).$$

- This cannot be simplified further.
- It provides our base-line.
- Oversampling is the desired behaviour!

## Undersampling, Aliasing

- When the sampling rate is such that $1 < \hat{f}_d \leq 3/2$, then we define the apparent frequency $\hat{f}_a = \hat{f}_d - 1$.
- Notice that $0 < \hat{f}_a \leq 1/2$ and $\hat{f}_d = \hat{f}_a + 1$.
  - For $f = 11$, $f_s = 10 \Rightarrow \hat{f}_d = 11/10 \Rightarrow \hat{f}_a = 1/10$.
- The samples of the sinusoidal signal are given by

$$x[n] = A\cos(2\pi\hat{f}_d n + \phi) = A\cos(2\pi(1 + \hat{f}_a)n + \phi).$$

- Expanding the terms inside the cosine,

$$x[n] = A\cos(2\pi\hat{f}_a n + 2\pi n + \phi) = A\cos(2\pi\hat{f}_a n + \phi)$$

- **Interpretation:** The samples are identical to those from a sinusoid with frequency $f = \hat{f}_a \cdot f_s$ and phase $\phi$.





## Undersampling, Folding

- When the sampling rate is such that $1/2 < \hat{f}_d \leq 1$, then we introduce the apparent frequency $\hat{f}_a = 1 - \hat{f}_d$; again $0 < \hat{f}_a \leq 1/2$; also $\hat{f}_d = 1 - \hat{f}_a$.
  - For $f = 9$, $f_s = 10 \Rightarrow \hat{f}_d = 9/10 \Rightarrow \hat{f}_a = 1/10$.
- The samples of the sinusoidal signal are given by

$$x[n] = A\cos(2\pi\hat{f}_d n + \phi) = A\cos(2\pi(1 - \hat{f}_a)n + \phi).$$

- Expanding the terms inside the cosine,

$$x[n] = A\cos(-2\pi\hat{f}_a n + 2\pi n + \phi) = A\cos(-2\pi\hat{f}_a n + \phi)$$

- Because of the symmetry of the cosine, this equals

$$x[n] = A\cos(2\pi\hat{f}_a n - \phi).$$

- **Interpretation:** The samples are identical to those from a sinusoid with frequency $f = \hat{f}_a \cdot f_s$ and phase $-\phi$ (phase reversal).

## Sampling Higher-Frequency Sinusoids

- For sinusoids of even higher frequencies $f$, either folding or aliasing occurs.
- As before, let $\hat{f}_d$ be the normalized frequency $f/f_s$.
- Decompose $\hat{f}_d$ into an integer part $N$ and fractional part $f_p$.
  - **Example:** If $\hat{f}_d$ is 5.7 then $N$ equals 5 and $f_p$ is 0.7.
  - Notice that $0 \leq f_p < 1$, always.
- **Phase Reversal** occurs when the phase of the sampled sinusoid is the negative of the phase of the continuous-time sinusoid.
- We distinguish between
  - **Folding** occurs when $f_p > 1/2$. Then the apparent frequency $\hat{f}_a$ equals $1 - f_p$ and phase reversal occurs.
  - **Aliasing** occurs when $f_p \leq 1/2$. Then the apparent frequency is $\hat{f}_a = f_p$; no phase reversal occurs.





## Examples

- For the three sinusoids considered earlier:
  1. $f = 1, \phi = \pi/4, f_s = 10 \Rightarrow \hat{f}_d = 1/10$
  2. $f = 9, \phi = -\pi/4, f_s = 10 \Rightarrow \hat{f}_d = 9/10$
  3. $f = 11, \phi = \pi/4, f_s = 10 \Rightarrow \hat{f}_d = 11/10$
- The first case, represents oversampling: The apparent frequency $\hat{f}_a = \hat{f}_d$ and no phase reversal occurs.
- The second case, represents folding: The apparent $\hat{f}_a$ equals $1 - \hat{f}_d$ and phase reversal occurs.
- In the final example, the fractional part of $\hat{f}_d = 1/10$. Hence, this case represents alising; no phase reversal occurs.

## Exercise

The discrete-time sinusoidal signal

$$x[n] = 5\cos(2\pi 0.2n - \frac{\pi}{4}).$$

was obtained by sampling a continuous-time sinusoid of the form

$$x(t) = A\cos(2\pi ft + \phi)$$

at the sampling rate $f_s = 8000$ *Hz*.

1. Provide three different sets of paramters $A$, $f$, and $\phi$ for the continuous-time sinusoid that all yield the discrete-time sinusoid above when sampled at the indicated rate. The parameter $f$ must satisfy $0 < f < 12000$ *Hz* in all three cases.
2. For each case indicate if the signal is undersampled or oversampled and if aliasing or folding occurred.





## Experiments

- Two experiments to illustrate the effects that sampling introduces:
  1. Sampling a chirp signal.
  2. Sampling a rotating phasor.

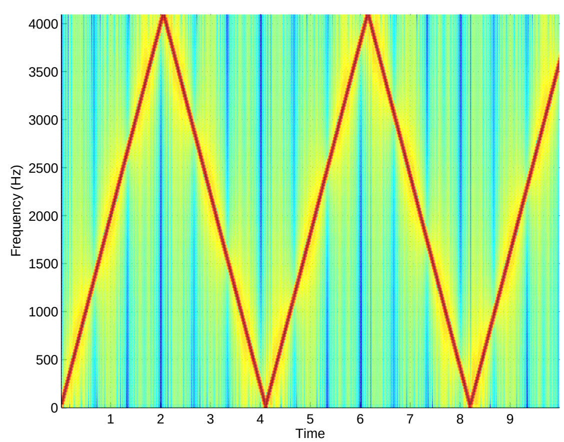
## Experiment: Sampling a Chirp Signal

- **Objective:** Directly observe folding and aliasing by means of a chirp signal.
- **Experiment Set-up:**
  - Set sampling rate. Baseline: $f_s = 44.1$KHz (oversampled), Comparison: $f_s = 8.192$KHz (undersampled)
  - Generate a (sampled) chirp signal with instantaneous frequency increasing from 0 to 20KHz in 10 seconds.
  - Evaluate resulting signal by
    - playing it through the speaker,
    - plotting the periodogram.
- **Expected Outcome?**
- **Expected Outcome:**
  - Directly observe folding and aliasing in second part of experiment.





## Periodogram of undersampled Chirp

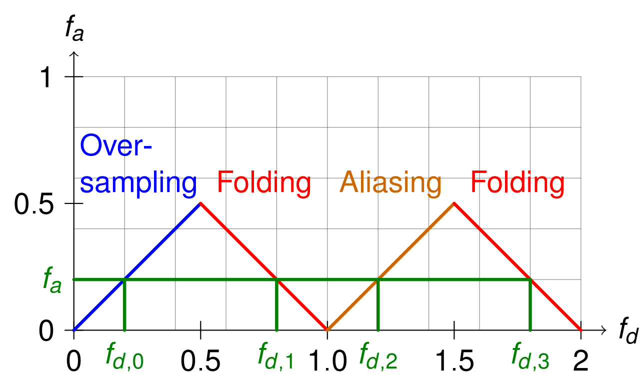


```
%% Parameters
fs = 8192;  % 44.1KHz for oversampling, 8192 for undersampling

% chitp: 0 to 20KHz in 10 seconds
fstart   = 0;
fend     = 20e3;
dur      = 10;

%% generate signal
tt   = 0:1/fs:dur;
psi  = 2*pi*(fend-fstart)/(2*dur)*tt.^2;  % phase function
xx   = cos(psi);

%% spectrogram
spectrogram( xx, 256, 128, 256, fs,'yaxis');

%% play sound
soundsc( xx, fs);
```





## Apparent and Normalized Frequency



## Experiment: Sampling a Rotating Phasor

- **Objective:** Investigate sampling effects when we can distinguish between positive and negative frequencies.
- **Experiment Set-up:**
  - Animation: rotating phasor in the complex plane.
  - Sampling rate describes the number of “snap-shots” per second (strobes).
  - Frequency the number of times the phasor rotates per second.
    - positive frequency: counter-clockwise rotation.
    - negative frequency: clockwise rotation.
- **Expected Outcome?**
- **Expected Outcome:**
  - Folding: leads to reversal of direction.
  - Aliasing: same direction but apparent frequency is lower than true frequency.





## True and Apparent Frequency

$f_s = 20$

| True Frequency | -0.5 | 0 | 0.5 | 19.5 | 20 | 20.5 |
|---|---|---|---|---|---|---|
| Apparent Frequency | -0.5 | 0 | 0.5 | -0.5 | 0 | 0.5 |

- Note, that instead of folding we observe negative frequencies.
  - occurs when true frequency equals 9.5 in above example.

```
%% parameters
fs  = 10;     % sampling rate in frames per second
dur = 10;     % signal duration in seconds

ff  = 9.5;  % frequency of rotating phasor
phi = 0;      % initial phase of phasor
A   = 1;      % amplitude

%% Prepare for plot
TitleString = sprintf('Rotating␣Phasor:␣f_d␣=␣%5.2f', ff/fs);
figure(1)

% unit circle (plotted for reference)
cc = exp(1j*2*pi*(0:0.01:1));
ccx = A*real(cc);
cci = A*imag(cc);
```





```
%% Animation
for tt = 0:1/fs:dur
    tic; % establish time-reference
    plot(ccx, cci, ':', ...
        [0 A*cos(2*pi*ff*tt+phi)], [0 A*sin(2*pi*ff*tt+phi)], '-ob');
    axis('square')
    axis([-A A -A A]);
    title(TitleString)
    xlabel('Real')
    ylabel('Imag')
    grid on;

    drawnow % force plots to be redrawn

    te = toc;

    % pause until the next sampling instant, if possible
    if ( te < 1/fs)
        pause(1/fs-te)
    end
end
```

## Lecture: The Sampling Theorem





## The Sampling Theorem

- We have analyzed the relationship between the frequency $f$ of a sinusoid and the sampling rate $f_s$.
  - We saw that the ratio $f/f_s$ must be less than $1/2$, i.e., $f_s > 2 \cdot f$. Otherwise aliasing or folding occurs.
- This insight provides the first half of the famous sampling theorem

> A continuous-time signal $x(t)$ with frequencies no higher than $f_{max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2 \cdot f_{max}$.

- This very import result is attributed to Claude Shannon and Harry Nyquist.

## Reconstructing a Signal from Samples

- The sampling theorem suggests that the original continuous-time signal $x(t)$ can be recreated from its samples $x[n]$.
  - Assuming that samples were taken at a high enough rate.
  - This process is referred to as reconstruction or D-to-C conversion (discrete-time to continuous-time conversion).
- In principle, the continous-time signal is reconstructed by placing a suitable pulse at each sample location and adding all pulses.
  - The amplitude of each pulse is given by the sample value.





## Suitable Pulses

- Suitable pulses include
  - Rectangular pulse (zero-order hold):

$$p(t) = \begin{cases} 1 & \text{for } -T_s/2 \leq t < T_s/2 \\ 0 & \text{else.} \end{cases}$$

  - Triangular pulse (linear interpolation)

$$p(t) = \begin{cases} 1 + t/T_s & \text{for } -T_s \leq t \leq 0 \\ 1 - t/T_s & \text{for } 0 \leq t \leq T_s \\ 0 & \text{else.} \end{cases}$$

## Reconstruction

- The reconstructed signal $\hat{x}(t)$ is computed from the samples and the pulse $p(t)$:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot p(t - nT_s).$$

- The reconstruction formula says:
  - place a pulse at each sampling instant ($p(t - nT_s)$),
  - scale each pulse to amplitude $x[n]$,
  - add all pulses to obtain the reconstructed signal.





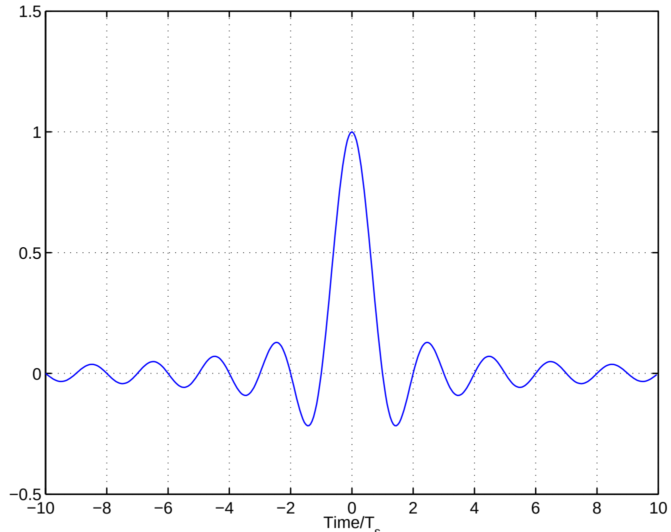
## Ideal Reconstruction

- Reconstruction with the above pulses will be pretty good.
  - Particularly, when the sampling rate is much greater than twice the signal frequency (significant oversampling).
- However, reconstruction is not perfect as suggested by the sampling theorem.
- To obtain perfect reconstruction the following pulse must be used:

$$p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}.$$

- This pulse is called the sinc pulse.
- Note, that it is of infinite duration and, therefore, is not practical.
  - In practice a truncated version may be used for excellent reconstruction.

## The sinc pulse








# Part V

## Introduction to Linear, Time-Invariant Systems

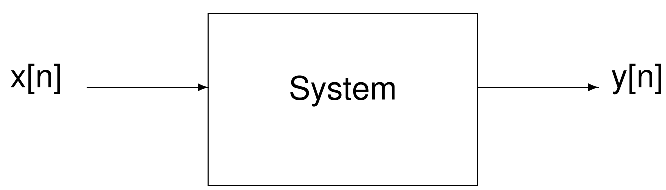
# Lecture: Introduction to Systems and FIR filters





## Systems

- A system is used to process an input signal $x[n]$ and produce the ouput signal $y[n]$.
  - We focus on discrete-time signals and systems;
  - a correspoding theory exists for continuous-time signals and systems.
- Many different systems:
  - Filters: remove undesired signal components,
  - Modulators and demodulators,
  - Detectors.



## Representative Examples

- The following are examples of systems:
  - **Squarer:** $y[n] = (x[n])^2$;
  - **Modulator:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$;
  - **Averager:** $y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$;
  - **FIR Filter:** $y[n] = \sum_{k=0}^{M} b_k x[n-k]$
- In MATLAB, systems are generally modeled as functions with $x[n]$ as the first input argument and $y[n]$ as the output argument.
  - **Example:** first two lines of function implementing a squarer.

```
function yy = squarer(xx)
% squarer - output signal is the square of the input signal
```





## Squarer

- System relationship between input and output signals:

$$y[n] = (x[n])^2.$$

- **Example:** Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
  - **Notation:** $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$ means $x[0] = 1$, $x[1] = 2$, ..., $x[6] = 1$; all other $x[n] = 0$.
- Output signal: $y[n] = \{1, 4, 9, 16, 9, 4, 1\}$.

## Modulator

- System relationship between input and output signals:

$$y[n] = (x[n]) \cdot \cos(2\pi f_d n);$$

  where the modulator frequency $f_d$ is a *parameter* of the system.
- **Example:**
  - Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
  - assume $f_d = 0.5$, i.e., $\cos(2\pi f_d n) = \{\ldots, 1, -1, 1, -1, \ldots\}$.
- Output signal: $y[n] = \{1, -2, 3, -4, 3, -2, 1\}$.





## Averager

- System relationship between input and output signals:

$$\begin{aligned} y[n] &= \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \\ &= \frac{1}{M} \cdot (x[n] + x[n-1] + \ldots + x[n-(M-1)]) \\ &= \sum_{k=0}^{M-1} \frac{1}{M} \cdot x[n-k]. \end{aligned}$$

- This system computes the *sliding average* over the $M$ most recent samples.
- **Example:** Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
- For computing the output signal, a table is very useful.
  - synthetic multiplication table.

## 3-Point Averager ($M = 3$)

| $n$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 0 |
| $\frac{1}{M} \cdot x[n]$ | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{4}{3}$ | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| $+\frac{1}{M} \cdot x[n-1]$ | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{4}{3}$ | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 |
| $+\frac{1}{M} \cdot x[n-2]$ | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{4}{3}$ | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ |
| $y[n]$ | 0 | $\frac{1}{3}$ | 1 | 2 | 3 | $\frac{10}{3}$ | 3 | 2 | 1 | $\frac{1}{3}$ |

- $y[n] = \{\frac{1}{3}, 1, 2, 3, \frac{10}{3}, 3, 2, 1, \frac{1}{3}\}$





## General FIR Filter

- The M-point averager is a special case of the general FIR filter.
  - FIR stands for Finite Impulse Response; we will see what this means later.
- The system relationship between the input $x[n]$ and the output $y[n]$ is given by

$$y[n] = \sum_{k=0}^{M-1} b_k \cdot x[n-k].$$

  - $M$ is the number of filter coefficients.
  - $M-1$ is called the order of the filter.

## General FIR Filter

- System relationship:

$$y[n] = \sum_{k=0}^{M-1} b_k \cdot x[n-k].$$

- The filter coefficients $b_k$ determine the characteristics of the filter.
  - Much more on the relationship between the filter coefficients $b_k$ and the characteristics of the filter later.
- Clearly, with $b_k = \frac{1}{M}$ for $k = 0, 1, \ldots, M-1$ we obtain the M-point averager.
- Again, computation of the output signal can be done via a synthetic multiplication table.
  - **Example:** $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$ and $b_k = \{1, -2, 1\}$.





## FIR Filter ($b_k = \{1, -2, 1\}$)

| $n$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 0 |
| $1 \cdot x[n]$ | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 0 |
| $-2 \cdot x[n-1]$ | 0 | 0 | -2 | -4 | -6 | -8 | -6 | -4 | -2 | 0 |
| $+1 \cdot x[n-2]$ | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 |
| $y[n]$ | 0 | 1 | 0 | 0 | 0 | -2 | 0 | 0 | 0 | 1 |

- $y[n] = \{1, 0, 0, 0, -2, 0, 0, 0, 1\}$
- Note that the output signal $y[n]$ is longer than the input signal $x[n]$.
- Note, synthetic multiplication works only for short, finite-duration signal.

## Exercise

1. Find the output signal $y[n]$ for an FIR filter

$$y[n] = \sum_{k=0}^{M-1} b_k \cdot x[n-k]$$

   with filter coefficients $b_k = \{1, -1, 2\}$ when the input signal is $x[n] = \{1, 2, 4, 2, 4, 2, 1\}$.



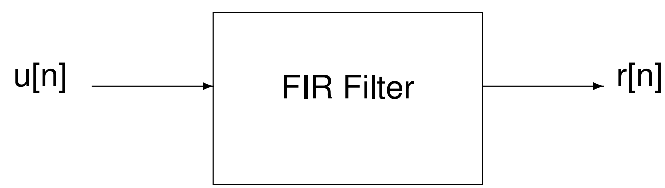


## Unit Step Sequence and Unit Step Response

- The signal with samples

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0 \end{cases}$$

  is called the unit-step sequence or unit-step signal.
- The output of an FIR filter when the input is the unit-step signal ($x[n] = u[n]$) is called the unit-step response $r[n]$.



## Unit-Step Response of the 3-Point Averager

- Input signal: $x[n] = u[n]$.
- Output signal: $r[n] = \frac{1}{3}\sum_{k=0}^{2} u[n-k]$.

| $n$ | -1 | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|---|
| $u[n]$ | 0 | 1 | 1 | 1 | 1 | ... |
| $\frac{1}{3}u[n]$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | ... |
| $+\frac{1}{3}u[n-1]$ | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | ... |
| $+\frac{1}{3}u[n-2]$ | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | ... |
| $r[n]$ | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | 1 | ... |





## Unit-Impulse Sequence and Unit-Impulse Response

- The signal with samples

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0 \end{cases}$$

  is called the unit-impulse sequence or unit-impulse signal.
- The output of an FIR filter when the input is the unit-impulse signal ($x[n] = \delta[n]$) is called the unit-impulse response, denoted $h[n]$.
- Typically, we will simply call the above signals simply impulse signal and impulse response.
- We will see that the impulse-response captures all characteristics of a FIR filter.
  - This implies that impulse response is a very important concept!

## Unit-Impulse Response of a FIR Filter

- Input signal: $x[n] = \delta[n]$.
- Output signal: $h[n] = \sum_{k=0}^{M-1} b_k \delta[n-k]$.

| $n$ | -1 | 0 | 1 | 2 | 3 | ... | M |
|---|---|---|---|---|---|---|---|
| $\delta[n]$ | 0 | 1 | 0 | 0 | 0 | ... | 0 |
| $b_0 \cdot \delta[n]$ | 0 | $b_0$ | 0 | 0 | 0 | ... | 0 |
| $+b_1 \cdot \delta[n-1]$ | 0 | 0 | $b_1$ | 0 | 0 | ... | 0 |
| $+b_2 \cdot \delta[n-2]$ | 0 | 0 | 0 | $b_2$ | 0 | ... | 0 |
| $\vdots$ | | | | $\vdots$ | | | |
| $+b_M \cdot \delta[n-M]$ | 0 | 0 | 0 | 0 | 0 | ... | $b_M$ |
| $h[n]$ | 0 | $b_0$ | $b_1$ | $b_2$ | $b_3$ | ... | $b_M$ |





## Important Insights

- For an FIR filter, the impulse response equals the sequence of filter coefficients:

$$h[n] = \begin{cases} b_n & \text{for } n = 0, 1, \ldots, M-1 \\ 0 & \text{else.} \end{cases}$$

- Because of this relationship, the system relationship for an FIR filter can also be written as

$$\begin{aligned} y[n] &= \sum_{k=0}^{M-1} b_k x[n-k] \\ &= \sum_{k=0}^{M-1} h[k] x[n-k] \\ &= \sum_{-\infty}^{\infty} h[k] x[n-k]. \end{aligned}$$

- The operation $y[n] = h[n] * x[n] = \sum_{-\infty}^{\infty} h[k] x[n-k]$ is called convolution; it is a **very, very** important operation.

## Exercise

1. Find the impulse response $h[n]$ for the FIR filter with difference equation

$$y[n] = 2 \cdot x[n] + x[n-1] - 3 \cdot x[n-3].$$

2. Compute the output signal, when the input signal is $x[n] = u[n]$.
3. Compute the output signal, when the input signal is $x[n] = \exp(-\alpha n) \cdot u[n]$.



Systems | Special Signals | Linear, Time-invariant Systems | Convolution and Linear, Time-invariant Systems | Implemen

Lecture: Linear, Time-Invariant Systems

## Introduction

- We have introduced systems as devices that process an input signal $x[n]$ to produce an output signal $y[n]$.
- **Example Systems:**
  - **Squarer:** $y[n] = (x[n])^2$
  - **Modulator:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$, with $0 < f_d \leq \frac{1}{2}$.
  - **FIR Filter:**

$$y[n] = \sum_{k=0}^{M-1} h[k] \cdot x[n-k].$$

    Recall that $h[k]$ is the impulse response of the filter and that the above operation is called convolution of $h[n]$ and $x[n]$.
- **Objective:** Define important characteristics of systems and determine which systems possess these characteristics.





## Causal Systems

- **Definition:** A system is called causal when it uses only the present and past samples of the input signal to compute the present value of the output signal.
- Causality is usually easy to determine from the system equation:
  - The output $y[n]$ must depend only on input samples $x[n], x[n-1], x[n-2], \ldots$.
  - Input samples $x[n+1], x[n+2], \ldots$ must not be used to find $y[n]$.
- **Examples:**
  - All three systems on the previous slide are causal.
  - The following system is non-causal:

$$y[n] = \frac{1}{3} \sum_{k=-1}^{1} x[n-k] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]).$$

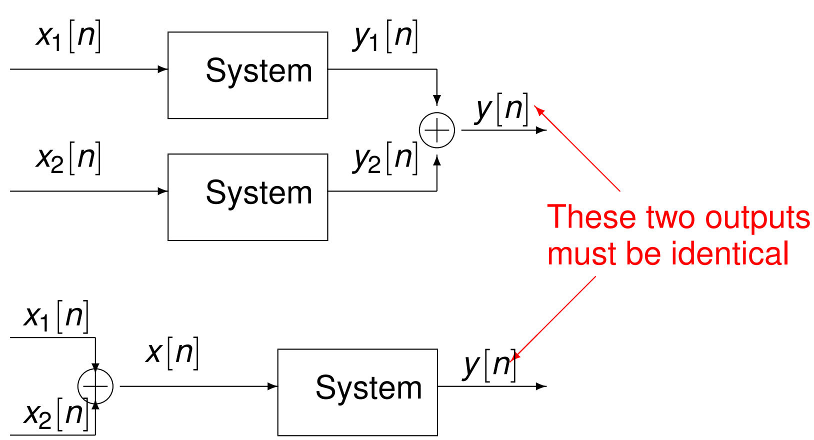
## Linear Systems

- The following test procedure defines linearity and shows how one can determine if a system is linear:
  1. **Reference Signals:** For $i = 1, 2$, pass input signal $x_i[n]$ through the system to obtain output $y_i[n]$.
  2. **Linear Combination:** Form a new signal $x[n]$ from the linear combination of $x_1[n]$ and $x_2[n]$:
     $$x[n] = x_1[n] + x_2[n].$$
     Then, Pass signal $x[n]$ through the system and obtain $y[n]$.
  3. **Check:** The system is linear if
     $$y[n] = y_1[n] + y_2[n]$$
- The above must hold for **all** inputs $x_1[n]$ and $x_2[n]$.
- For a linear system, the superposition principle holds.



Systems Special Signals Linear, Time-invariant Systems Convolution and Linear, Time-invariant Systems Implemen

## Illustration



## Example: Squarer

- **Squarer:** $y[n] = (x[n])^2$
  1. **References:** $y_i[n] = (x_i[n])^2$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and
  $$\begin{aligned} y[n] &= (x[n])^2 = (x_1[n] + x_2[n])^2 \\ &= (x_1[n])^2 + (x_2[n])^2 + 2x_1[n]x_2[n]. \end{aligned}$$
  3. **Check:**
  $$y[n] \neq y_1[n] + y_2[n] = (x_1[n])^2 + (x_2[n])^2.$$
- **Conclusion:** not linear.





## Example: Modulator

- **Modulator:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$
  1. **References:** $y_i[n] = x_i[n] \cdot \cos(2\pi f_d n)$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and
  $$\begin{aligned} y[n] &= x[n] \cdot \cos(2\pi f_d n) \\ &= (x_1[n] + x_2[n]) \cdot \cos(2\pi f_d n). \end{aligned}$$
  3. **Check:**
  $$y[n] = y_1[n] + y_2[n] = x_1[n] \cdot \cos(2\pi f_d n) + x_2[n] \cdot \cos(2\pi f_d n).$$
- **Conclusion:** linear.

## Example: FIR Filter

- **FIR Filter:** $y[n] = \sum_{k=0}^{M-1} h[k] \cdot x[n-k]$
  1. **References:** $y_i[n] = \sum_{k=0}^{M-1} h[k] \cdot x_i[n-k]$ for $i = 1, 2$.
  2. **Linear Combination:** $x[n] = x_1[n] + x_2[n]$ and

$$y[n] = \sum_{k=0}^{M-1} h[k] \cdot x[n-k] = \sum_{k=0}^{M-1} h[k] \cdot (x_1[n-k] + x_2[n-k]).$$

  3. **Check:**

$$y[n] = y_1[n] + y_2[n] = \sum_{k=0}^{M-1} h[k] \cdot x_1[n-k] + \sum_{k=0}^{M-1} h[k] \cdot x_2[n-k].$$

- **Conclusion:** linear.





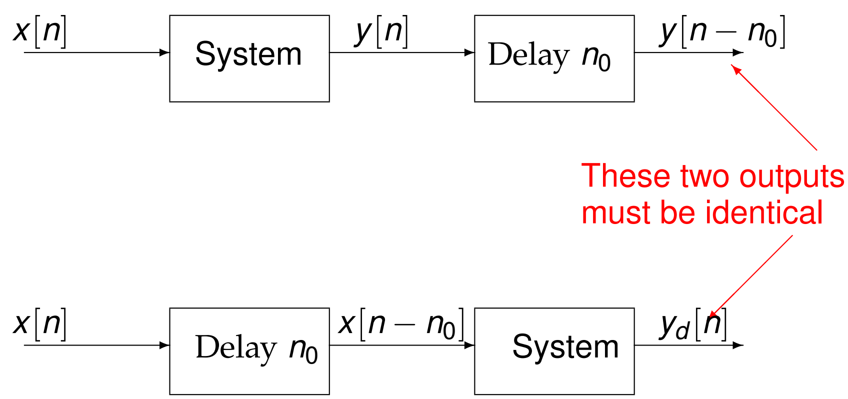
## Time-invariance

- The following test procedure defines time-invariance and shows how one can determine if a system is time-invariant:
  1. **Reference:** Pass input signal $x[n]$ through the system to obtain output $y[n]$.
  2. **Delayed Input:** Form the delayed signal $x_d[n] = x[n - n_0]$. Then, Pass signal $x_d[n]$ through the system and obtain $y_d[n]$.
  3. **Check:** The system is time-invariant if

$$y[n - n_0] = y_d[n]$$

- The above must hold for **all** inputs $x[n]$ and all delays $n_0$.
- **Interpretation:** A time-invariant system does not change, over time, the way it processes the input signal.

## Illustration








## Example: Squarer

- **Squarer:** $y[n] = (x[n])^2$
  1. **Reference:** $y[n] = (x[n])^2$.
  2. **Delayed Input:** $x_d[n] = x[n - n_0]$ and
  $$y_d[n] = (x_d[n])^2 = (x[n - n_0])^2.$$
  3. **Check:**
  $$y[n - n_0] = (x[n - n_0])^2 = y_d[n].$$
- **Conclusion:** time-invariant.

## Example: Modulator

- **Modulator:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$.
  1. **Reference:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$.
  2. **Delayed Input:** $x_d[n] = x[n - n_0]$ and
  $$y_d[n] = x_d[n] \cdot \cos(2\pi f_d n) = x[n - n_0] \cdot \cos(2\pi f_d n).$$
  3. **Check:**
  $$y[n - n_0] = x[n - n_0] \cdot \cos(2\pi f_d (n - n_0)) \neq y_d[n].$$
- **Conclusion:** not time-invariant.





## Example: Modulator

- Alternatively, to show that the modulator is **not** time-invariant, we construct a counter-example.
- Let $x[n] = \{0, 1, 2, 3, \ldots\}$, i.e., $x[n] = n$, for $n \geq 0$.
- Also, let $f_d = \frac{1}{2}$, so that
$$\cos(2\pi f_d n) = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases}$$
- Then, $y[n] = x[n] \cdot \cos(2\pi f_d n) = \{0, -1, 2, -3, \ldots\}$.
- With $n_0 = 1$, $x_d[n] = x[n-1] = \{0, 0, 1, 2, 3, \ldots\}$, we get $y_d[n] = \{0, 0, 1, -2, 3, \ldots\}$.
- Clearly, $y_d[n] \neq y[n-1]$.
- not time-invariant

## Example: FIR Filter

- **Reference:** $y[n] = \sum_{k=0}^{M-1} h[k] \cdot x[n-k]$.
- **Delayed Input:** $x_d[n] = x[n-n_0]$, and

$$y_d[n] = \sum_{k=0}^{M-1} h[k] \cdot x_d[n-k] = \sum_{k=0}^{M-1} h[k] \cdot x[n-n_0-k].$$

- **Check:**

$$y[n-n_0] = \sum_{k=0}^{M-1} h[k] \cdot x[n-n_0-k] = y_d[n]$$

- time-invariant





## Exercise

- Let $u[n]$ be the unit-step sequence (i.e., $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$, otherwise).
- The system is a 3-point averager:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]).$$

1. Find the output $y_1[n]$ when the input $x_1[n] = u[n]$.
2. Find the output $y_2[n]$ when the input $x_2[n] = u[n-2]$.
3. Find the output $y[n]$ when the input $x[n] = u[n] - u[n-2]$.
4. How are linearity and time-invariance evident in your results?

# Lecture: Convolution and Linear, Time-Invariant Systems





## Overview

- **Today:** a really important, somewhat challenging, class.
- **Key result:** for every linear, time-invariant system (LTI system) the output is obtained from input via convolution.
  - Convolution is a very important operation!
- Prerequisites from previous classes:
  - Impulse signal and impulse response,
  - convolution,
  - linearity, and
  - time-invariance.

## Reminders: Convolution and Impulse Response

- **We learned so far:**
  - For FIR filters, input-output relationship

$$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

  - If $x[n] = \delta[n]$, then $y[n] = h[n]$ is called the impulse response of the system.
    - For FIR filters:

$$h[n] = \begin{cases} b_n & \text{for } 0 \le n \le M \\ 0 & \text{else.} \end{cases}$$

  - **Convolution:** input-output relationship

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$





## Reminders: Linearity and Time-Invariance

- **Linearity:**
  - For arbitrary input signals $x_1[n]$ and $x_2[n]$, let the ouputs be denoted $y_1[n]$ and $y_2[n]$.
  - Further, for the input signal $x[n] = x_1[n] + x_2[n]$, let the output signal be $y[n]$.
  - The system is linear if $y[n] = y_1[n] + y_2[n]$.
- **Time-Invariance:**
  - For an arbitrary input signal $x[n]$, let the output be $y[n]$.
  - For the delayed input $x_d[n] = x[n-n_0]$, let the output be $y_d[n]$.
  - The system is time-invariant if $y_d[n] = y[n-n_0]$.
- **Today:** For any linear, time-invariant system: input-output relationship is $y[n] = x[n] * h[n]$.

## Preliminaries

- We need a few more facts and relationships for the impulse signal $\delta[n]$.
- To start, recall:
  - If input to a system is the impulse signal $\delta[n]$,
  - then, the output is called the impulse response,
  - and is denoted by $h[n]$.
- We will derive a method for expressing arbitrary signals $x[n]$ in terms of impulses.





## Sifting with Impulses

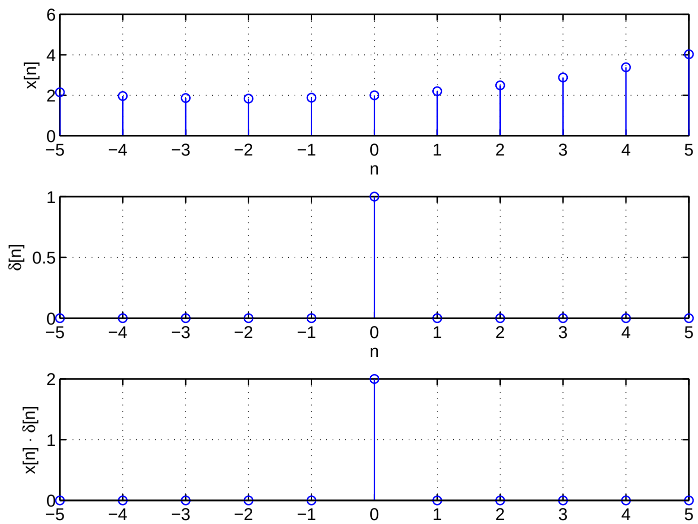
- **Question:** What happens if we multiply a signal $x[n]$ with an impulse signal $\delta[n]$?
- Because

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{else,} \end{cases}$$

- it follows that

$$x[n] \cdot \delta[n] = x[0] \cdot \delta[n] = \begin{cases} x[0] & \text{for } n = 0 \\ 0 & \text{else} \end{cases}$$

## Illustration








## Sifting with Impulses

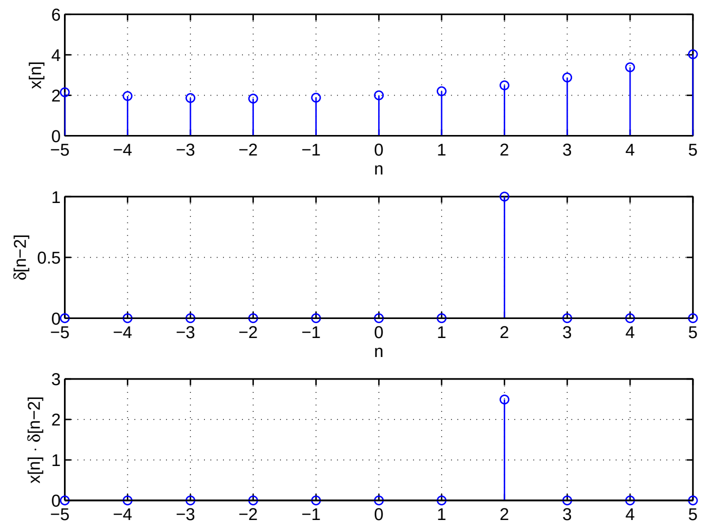
- **Related Question:** What happens if we multiply a signal $x[n]$ with a delayed impulse signal $\delta[n-k]$?
- Recall that $\delta[n-k]$ is an impulse located at the $k$-th sampling instance:

$$\delta[n-k] = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{else} \end{cases}$$

- It follows that

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k] = \begin{cases} x[k] & \text{for } n = k \\ 0 & \text{else} \end{cases}$$

## Illustration








## Decomposing a Signal with Impulses

- **Question:** What happens if we combine (add) signals of the form $x[n] \cdot \delta[n-k]$?
- Specifically, what is

$$\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]?$$

  - Notice that the above sum represents the convolution of $x[n]$ and $\delta[n]$, $\delta[n] * x[n]$.

## Decomposing a Signal with Impulses

| $n$ | ... | -1 | 0 | 1 | 2 | ... |
|---|---|---|---|---|---|---|
| $x[n]$ | ... | x[-1] | x[0] | x[1] | x[2] | ... |
| $\delta[n]$ | ... | 0 | 1 | 0 | 0 | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x[-1] \cdot \delta[n+1]$ | ... | x[-1] | 0 | 0 | 0 | ... |
| $x[0] \cdot \delta[n]$ | ... | 0 | x[0] | 0 | 0 | ... |
| $x[1] \cdot \delta[n-1]$ | ... | 0 | 0 | x[1] | 0 | ... |
| $x[2] \cdot \delta[n-2]$ | ... | 0 | 0 | 0 | x[2] | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$ | ... | x[-1] | x[0] | x[1] | x[2] | ... |





## Decomposing a Signal with Impulses

- From these considerations we conclude that

$$\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] = x[n].$$

  - Notice that this implies

$$x[n] * \delta[n] = x[n].$$

- We now have a way to write a signal $x[n]$ as a sum of scaled and delayed impulses.
- Next, we exploit this relationship to derive our main result.

## Applying Linearity and Time-Invariance

- We know already that input $\delta[n]$ produces output $h[n]$ (impulse repsonse). We write:

$$\delta[n] \mapsto h[n].$$

- For a time-invariant system:

$$\delta[n-k] \mapsto h[n-k].$$

- And for a linear system:

$$x[k] \cdot \delta[n-k] \mapsto x[k] \cdot h[n-k].$$





## Derivation of the Convolution Sum

- Linearity: linear combination of input signals produces output equal to linear combination of individual outputs.

| Input | $\mapsto$ | Output |
|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ |
| $x[-1] \cdot \delta[n+1]$ | $\mapsto$ | $x[-1] \cdot h[n+1]$ |
| $x[0] \cdot \delta[n]$ | $\mapsto$ | $x[0] \cdot h[n]$ |
| $x[1] \cdot \delta[n-1]$ | $\mapsto$ | $x[1] \cdot h[n-1]$ |
| $x[2] \cdot \delta[n-1]$ | $\mapsto$ | $x[2] \cdot h[n-2]$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] = x[n]$ | $\mapsto$ | $y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$ |

## Summary and Conclusions

- We just derived the convolution sum formula:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$

- We only assumed that the system is linear and time-invariant.
- Therefore, we can conclude that *for any* linear, time-invariant system, the output is the convolution of input and impulse response.
  - Needless to say: convolution and impulse response are enormously important concepts.





## Identity System

- From our discussion, we can draw another conclusion.
- **Question:** How can we characterize a LTI system for which the output $y[n]$ is the same as the input $x[n]$.
  - Such a system is called the identity system.
- Specifically, we want the impulse response $h[n]$ of such a system.
- As always, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input.
- Since the ouput is the same as the input for an identity system, we find the impulse response of the identity system

$$h[n] = \delta[n].$$

## Ideal Delay Systems

- **Closely Related Question:** How can one characterize a LTI system for which the output $y[n]$ is a delayed version of the input $x[n]$:
$$y[n] = x[n - n_0]$$
where $n_0$ is the delay introduced by the system
  - Such a system is called an ideal delay system.
- Again, we want the impulse response $h[n]$ of such a system.
- As before, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input.
- Since the ouput is merely a delayed version of the input, we find
$$h[n] = \delta[n - n_0].$$





## Exercise

- Show that convolution is a commutative operation, i.e., that $x[n] * h[n]$ equals $h[n] * x[n]$.

## Lecture: Convolution and Linear, Time-Invariant Systems



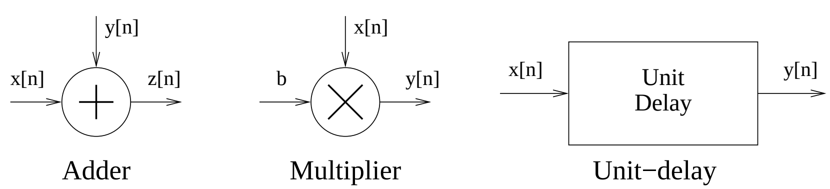


## Building Blocks

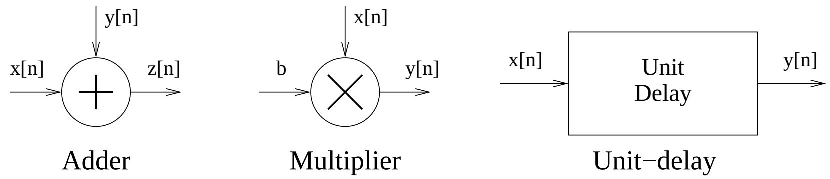
- Recall that the input-output relationship for an FIR filter is given by

$$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

- Digital systems implementing this relationships are easily constructed from simple building blocks:



## Operation of Building Blocks




- **Adder:** sum of two signals
$$z[n] = x[n] + y[n].$$
- **Multiplier:** product of signal with a scalar
$$y[n] = b \cdot x[n]$$
- **Unit-delay:** delays input by one sample:
$$y[n] = x[n-1]$$





## Block Diagrams

# Part VI

## Frequency Response

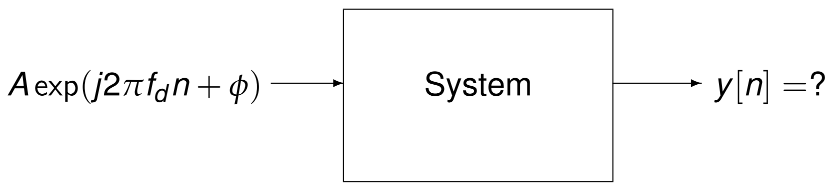


Introduction to Frequency Response
Frequency Response of LTI Systems
A comprehensive Example

## Lecture: Introduction to Frequency Response

## Introduction

- We have discussed:
  - Sinusoidal and complex exponential signals,
  - Spectrum representation of signals:
    - arbitrary signals can be expressed as the sum of sinusoidal (or complex exponential) signals.
  - Linear, time-invariant systems.
- Next: complex exponential signals as input to linear, time-invariant systems.








## Example: 3-Point Averaging Filter

- Consider the 3-point averager:

$$y[n] = \frac{1}{3} \sum_{k=0}^{2} x[n-k] = \frac{1}{3} \cdot (x[n] + x[n-1] + x[n-2]).$$

- **Question:** What is the output $y[n]$ if the input is $x[n] = \exp(j2\pi f_d n)$?
  - Recall that $f_d$ is the normalized frequency $f/f_s$; we are assuming the signal is oversampled, $|f_d| < \frac{1}{2}$
  - Initially, assume $A = 1$ and $\phi = 0$; generalization is easy.

## Delayed Complex Exponentials

- The 3-point averager involves delayed versions of the input signal.
- We begin by assessing the impact the delay has on the complex exponential input signal.
- For
$$x[n] = \exp(j2\pi f_d n)$$
a delay by $k$ samples leads to
$$\begin{aligned} x[n-k] &= \exp(j2\pi f_d(n-k)) \\ &= e^{j(2\pi f_d n - 2\pi f_d k)} = e^{j2\pi f_d n} \cdot e^{-j2\pi f_d k} \\ &= e^{j(2\pi f_d n + \phi_k)} = e^{j2\pi f_d n} \cdot e^{j\phi_k} \end{aligned}$$
where $\phi_k = -2\pi f_d k$ is the phase shift induced by the $k$ sample delay.





## Average of Delayed Complex Exponentials

- Now, the output signal $y[n]$ is the average of three delayed complex exponentials
$$\begin{aligned} y[n] &= \frac{1}{3}\sum_{k=0}^{2} x[n-k] \\ &= \frac{1}{3}\sum_{k=0}^{2} e^{j(2\pi f_d n - 2\pi f_d k)} \end{aligned}$$
- This expression involves the sum of complex exponentials of the same frequency; the phasor addition rule applies:
$$y[n] = e^{j2\pi f_d n} \cdot \frac{1}{3}\sum_{k=0}^{2} e^{-j2\pi f_d k}.$$
- **Important Observation:** The output signal is a complex exponential of the same frequency as the input signal.
  - The amplitude and phase are different.

## Frequency Response of the 3-Point Averager

- The output signal $y[n]$ can be rewritten as:

$$\begin{aligned} y[n] &= e^{j2\pi f_d n} \cdot \tfrac{1}{3}\sum_{k=0}^{2} e^{-j2\pi f_d k} \\ &= e^{j2\pi f_d n} \cdot H(e^{j2\pi f_d}). \end{aligned}$$

where

$$\begin{aligned} H(e^{j2\pi f_d}) &= \tfrac{1}{3}\sum_{k=0}^{2} e^{-j2\pi f_d k} \\ &= \tfrac{1}{3} \cdot (1 + e^{-j2\pi f_d} + e^{-j2\pi 2 f_d}) \\ &= \tfrac{1}{3} \cdot e^{-j2\pi f_d}(e^{j2\pi f_d} + 1 + e^{-j2\pi f_d}) \\ &= \tfrac{e^{-j2\pi f_d}}{3}(1 + 2\cos(2\pi f_d)). \end{aligned}$$



Introduction to Frequency Response · Frequency Response of LTI Systems · A comprehensive Example

## Interpretation

- From the above, we can conclude:
  - If the input signal is of the form $x[n] = \exp(j2\pi f_d n)$,
  - then the output signal is of the form $y[n] = H(e^{j2\pi f_d}) \cdot \exp(j2\pi f_d n)$.
- The function $H(e^{j2\pi f_d})$ is called the frequency response of the system.
- **Note:** If we know $H(e^{j2\pi f_d})$, we can easily compute the output signal in response to a complex expontial input signal.

## Examples

- Recall:

$$H(e^{j2\pi f_d}) = \frac{e^{-j2\pi f_d}}{3}(1 + 2\cos(2\pi f_d))$$

- Let $x[n]$ be a complex exponential with $f_d = 0$.
  - Then, all samples of $x[n]$ equal to one.
- The output signal $y[n]$ also has all samples equal to one.
- For $f_d = 0$, the frequency response $H(e^{j2\pi 0}) = 1$.
- And, the output $y[n]$ is given by

$$y[n] = H(e^{j2\pi 0}) \cdot \exp(j2\pi 0 n),$$

i.e., all samples are equal to one.





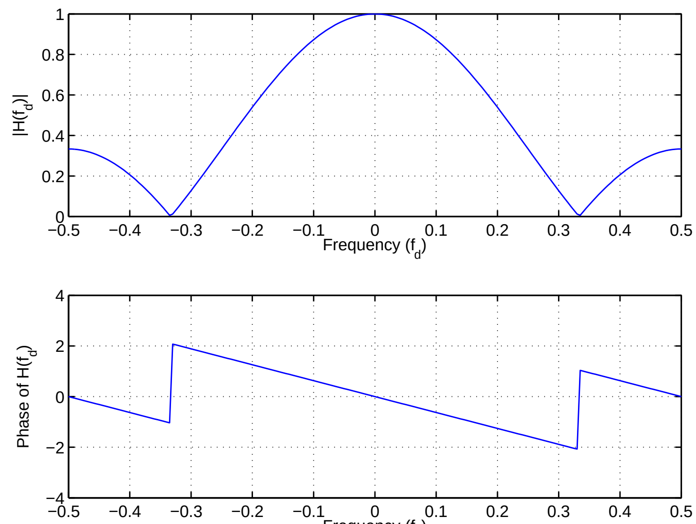
## Examples

- Let $x[n]$ be a complex exponential with $f_d = \frac{1}{3}$.
  - Then, the samples of $x[n]$ are the periodic repetition of $\{1, -\frac{1}{2} + \frac{j\sqrt{3}}{2}, -\frac{1}{2} - \frac{j\sqrt{3}}{2}\}$.
- The 3-point average over three consecutive samples equals zero; therefore, $y[n] = 0$.
- For $f_d = \frac{1}{3}$, the frequency response $H(e^{j2\pi f_d}) = 0$.
- Consequently, the output $y[n]$ is given by

$$y[n] = H(\frac{1}{3}) \cdot \exp(j2\pi \frac{1}{3} n) = 0.$$

Thus, all output samples are equal to zero.

## Plot of Frequency Response








## General Complex Exponential

- Let $x[n]$ be a complex exponential of the from $Ae^{j(2\pi f_d n+\phi)}$.
  - This signal can be written as

$$x[n] = X \cdot e^{j2\pi f_d n},$$

    where $X = Ae^{j\phi}$ is the *phasor* of the signal.
- Then, the output $y[n]$ is given by

$$y[n] = H(e^{j2\pi f_d}) \cdot X \cdot \exp(j2\pi f_d n).$$

- **Interpretation:** The output is a complex exponential of the same frequency $f_d$
- The phasor for the output signal is the product $H(e^{j2\pi f_d}) \cdot X$.

## Exercise

Assume that the signal $x[n] = \exp(j2\pi f_d n)$ is input to a 4-point averager.

1. Give a general expression for the output signal and identify the frequenchy response of the system.
2. Compute the output signals for the specific frequencies $f_d = 0$, $f_d = 1/4$, and $f_d = 1/2$.





# Lecture: The Frequency Response of LTI Systems

## Introduction

- We have demonstrated that for linear, time-invariant systems
  - the output signal $y[n]$
  - is the convolution of the input signal $x[n]$ and the impulse response $h[n]$.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=0}^{M} h[k] \cdot x[n-k] \end{aligned}$$

- **Question:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi f_d n + \phi))$.





## Response to a Complex Exponential

- **Problem:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi f_d n + \phi))$.
- Output $y[n]$ is convolution of input and impulse response

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=0}^{M} h[k] \cdot x[n-k] \\ &= \sum_{k=0}^{M} h[k] \cdot A\exp(j(2\pi f_d (n-k) + \phi)) \\ &= A\exp(j(2\pi f_d n + \phi)) \cdot \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi f_d k) \\ &= A\exp(j(2\pi f_d n + \phi)) \cdot H(e^{j2\pi f_d}) \end{aligned}$$

- The term

$$H(e^{j2\pi f_d}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi f_d k)$$

is called the Frequency Response of the system.

## Interpreting the Frequency Response

The Frequency Response of an LTI system with impulse response $h[n]$ is

$$H(e^{j2\pi f_d}) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi f_d k)$$

- **Observations:**
  - The response of a LTI system to a complex exponential signal is a complex exponential signal of the same frequency.
    - Complex exponentials are eigenfunctions of LTI systems.
  - When $x[n] = A\exp(j(2\pi f_d n + \phi))$, then $y[n] = x[n] \cdot H(e^{j2\pi f_d})$.
    - This is true only for complex exponential input signals!





## Interpreting the Frequency Response

- **Observations:**
  - $H(e^{j2\pi f_d})$ is best interpreted in polar coordinates:

$$H(e^{j2\pi f_d}) = |H(e^{j2\pi f_d})| \cdot e^{j\angle H(e^{j2\pi f_d})}.$$

    - Then, for $x[n] = A\exp(j(2\pi f_d n + \phi))$

$$\begin{aligned} y[n] &= x[n] \cdot H(e^{j2\pi f_d}) \\ &= A\exp(j(2\pi f_d n + \phi)) \cdot |H(e^{j2\pi f_d})| \cdot e^{j\angle H(e^{j2\pi f_d})} \\ &= (A \cdot |H(e^{j2\pi f_d})|) \cdot \exp(j(2\pi f_d n + \phi + \angle H(e^{j2\pi f_d}))) \end{aligned}$$

  - The amplitude of the resulting complex exponential is the product $A \cdot |H(e^{j2\pi f_d})|$.
    - Therefore, $|H(e^{j2\pi f_d})|$ is called the gain of the system.
  - The phase of the resulting complex exponential is the sum $\phi + \angle H(e^{j2\pi f_d})$.
    - $\angle H(e^{j2\pi f_d})$ is called the phase of the system.

## Example

- Let $h[n] = \{1, -2, 1\}$.
- Then,

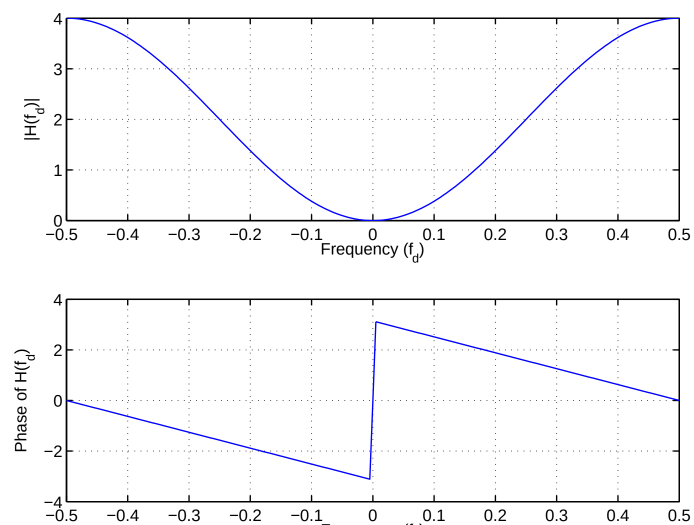
$$\begin{aligned} H(e^{j2\pi f_d}) &= \sum_{k=0}^{2} h[k] \cdot \exp(-j2\pi f_d k) \\ &= 1 - 2 \cdot \exp(-j2\pi f_d) + 1 \cdot \exp(-j2\pi f_d 2) \\ &= \exp(-j2\pi f_d) \cdot (\exp(j2\pi f_d) - 2 + \exp(-j2\pi f_d)) \\ &= \exp(-j2\pi f_d) \cdot (2\cos(2\pi f_d) - 2). \end{aligned}$$

- Gain: $|H(e^{j2\pi f_d})| = |2\cos(2\pi f_d) - 2|$





## Example



## Example

- The filter with impulse response $h[n] = \{1, -2, 1\}$ is a high-pass filter.
  - It rejects sinusoids with frequencies near $f_d = 0$,
  - and passes sinusoids with frequencies near $f_d = \frac{1}{2}$
- Note how the function of this system is much easier to describe in terms of the frequency response $H(e^{j2\pi f_d})$ than in terms of the impulse response $h[n]$.
- **Question:** Find the output signal when input equals $x[n] = 2\exp(j2\pi 1/4n - \pi/2)$.
- **Solution:**

$$H(\frac{1}{4}) = \exp(-j2\pi\frac{1}{4}) \cdot (2\cos(2\pi\frac{1}{4}) - 2) = -2e^{-j\pi/2} = 2e^{j\pi/2}.$$

Thus,

$$y[n] = 2e^{j\pi/2} \cdot x[n] = 4\exp(j2\pi n/4).$$





## Exercise

1. Find the Frequency Response $H(e^{j2\pi f_d})$ for the LTI system with impulse response $h[n] = \{1, -1, -1, 1\}$.
2. Find the output for the input signal $x[n] = 2\exp(j(2\pi n/3 - \pi/4))$.

## Computing Frequency Response in MATLAB

```
function HH = FreqResp( hh, ff )
% FreqResp - compute frequency response of LTI system
%
% inputs:
%     hh - vector of impulse repsonse coefficients
%     ff - vector of frequencies at which to evaluate frequency respon
%
% output:
%     HH - frequency response at frequencies in ff.
%
% Syntax:
%     HH = FreqResp( hh, ff )

HH = zeros( size(ff) );
for kk = 1:length(hh)
    HH = HH + hh(kk)*exp(-j*2*pi*(kk-1)*ff);
end
```





## Lecture: Comprehensive Example

## Introduction

- **Objective:** Apply many of the things we covered to the solution of a "real-world" problem.
- **Problem:** Design and implement a decoder for "touch-tone" dialing.
- When dialing a digit on a telphone touch-pad a two-tone signal is emitted. These are called dual tone multifrequency (DTMF) signals.

| Frequencies (Hz) | 1209 | 1336 | 1477 |
|---|---|---|---|
| 697 | 1 | 2 | 3 |
| 770 | 4 | 5 | 6 |
| 852 | 7 | 8 | 9 |
| 941 | * | 0 | # |





## Generating DTMF Signals

- Generating DTMF signals for a given digit is straightforward.
  - Determine the frequencies that the signal contains,
  - Generate two sinusoids of these frequencies,
  - Add sinusoids.
- Repeat for each digit to be dialed.
- The following MATLAB code extracts digits to be dialed from a string and forms the signal.
- Function signature:

```
function tones = dtmfdial( string, fs, tonedur, pausedur)
```

## Parsing the Dial-String

```
%% lookup table to translate digits string into numbers
Digits = double('123456789*0#');
InverseDigits = zeros(1,length(Digits) );
for kk=1:12
    InverseDigits( Digits(kk) ) = kk;
end

RawNumbers = double( string );
numbers = InverseDigits( RawNumbers );

% ensure numbers are integers between 1 and 12
numbers = round( numbers ); % silently discard fractional part
if ( min( numbers ) < 1 || max( numbers ) > 12 )
    error( 'input␣numbers␣must␣be␣integers␣between␣1␣and␣12' );
end
```





## Generating the DTMF Signal

```
%% construct signal
% convert durations to number of samples
Ntone  = round( fs*tonedur );
Npause = round( fs*pausedur);

% figure out how long the output signal will be
Nnumbers = length( numbers );
Nsamples = Nnumbers*(Ntone + Npause);

tones = zeros(1, Nsamples );
pause = zeros(1, Npause);

% associate numbers with DTMF pairs, record normalized frequencies!
dtmfpairs = ...
    [ 697 697 697 770 770 770 852 852 852 941 941 941;
      1209 1336 1477 1209 1336 1477 1209 1336 1477 1209 1336 1477 ]/fs
```

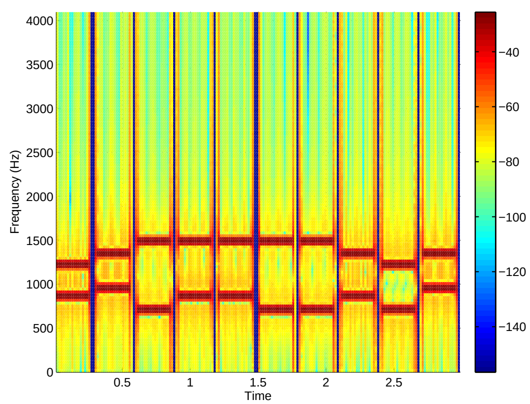
# Generating the DTMF Signal

```
% loop over all numbers
for kk = 1:length(numbers)
    Start = (kk-1)*(Ntone + Npause) + 1;
    End   = kk*(Ntone + Npause);

    freqs = dtmfpairs( :, numbers(kk) );
    currtone = 0.5* cos( 2*pi*freqs(1)*(0:Ntone-1) ) + ...
        0.5*cos( 2*pi*freqs(2)*(0:Ntone-1) );
    tones(Start:End) = [ currtone pause ];
end
```





# Spectrogram of Signal



## Plan for Recovering the Dial String

- Use bandpass-filters for each of the possible frequencies
  - **Intent:** Isolate the different tones.
- Detect the strongest two tones in each dialing period.
- Map tones to digits (decoding)





## A simple bandpass filter

- We discussed the $M$-point averager and showed that it has low-pass filter characteristics.
  - Note that the averager's impulse response consists of $M$ samples of a constant signal.
- Analogously, a simple bandpass filter centered at frequency $f_0$ has impulse response equal to
  - $M$ samples of $2/M \cos(2\pi f_0 n)$.
- The following MATLAB function implements this design strategy.
  - Alternatively, we could use MATLAB's filter design tools.

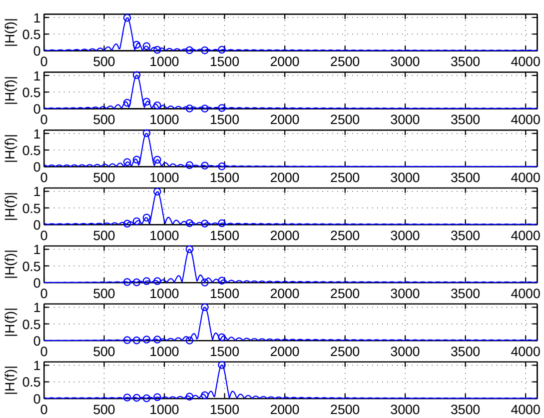
# MATLAB function makeBPF.m

```
function hh = makeBPF( fd, N )
% makeBPF - design simple bandpass filter
%
% usage:
%   hh = makeBPF( fd, N )
%
% inputs:
%   fd - center frequency of pass band (normalized by fs)
%   N  - number of filter coefficients
%
% output:
%   hh - vector of filter coefficients

% sample locations
nn = -(N-1)/2:1:(N-1)/2;

% impulse response
hh = 2/N*cos(2*pi*fd*nn);
```

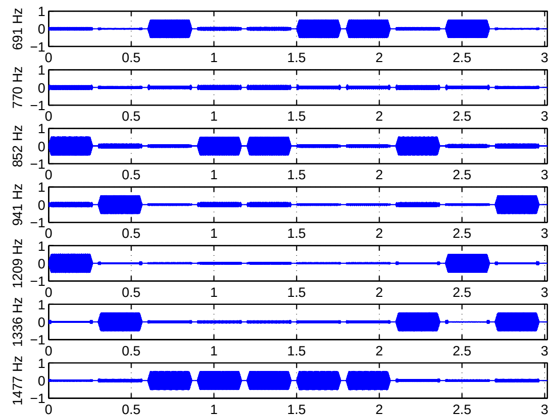




# Frequency Response of Bandpass Filters



## Output of Bandpass Filters








## Detecting Tones

- The presence or absence is fairly easy to see in the output of the bandpass filters.
- However a single metric is needed to determine the presence or absence of each tone.
- Good strategy: for each filter output $k = 1, \ldots, 7$ and each dialing-period $m = 1, \ldots, 10$, compute the following score s

$$s(k, m) = \sum_{n \text{ in } m\text{-th dialing period}} (y_k[n])^2,$$

where $y_k$ denotes the output of the $k$-th bandpass filter.

  - Note that this operation assumes that we know exactly where each digit starts.
- MATLAB code for computing scores follows.

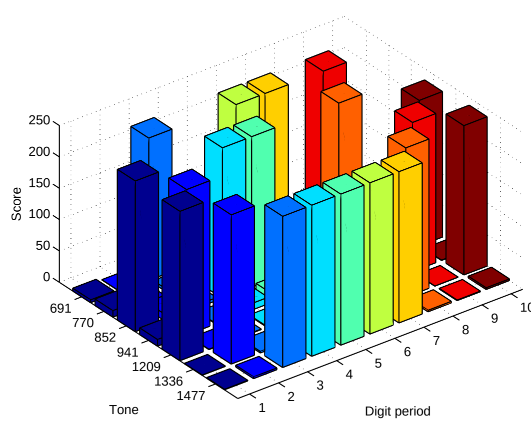
# MATLAB code for Computing Scores

```
pause

% decision logic
% decompose samples into periods for each number
Nnumbers = floor( length(xx)/(fs*(tonedur+pausedur)) );
NTonePlusPause = round(fs*(tonedur+pausedur));
NPause = round(fs*pausedur);

% score for each tone period: sum of squares in period
score = zeros(Nnumbers, length(DTMFFreqs));
for nn=1:Nnumbers
    Startnn = (nn-1)*NTonePlusPause + 1 + floor(LBPF/2);
    Endnn = nn*NTonePlusPause - NPause - floor(LBPF/2);
    for kk = 1:length(DTMFFreqs)
```





# Scores



## Decoding

- It remains to find the two highest scores in each dialing period.
  - More specifically, the highest score among the lower four frequencies and the highest score among the higher three frequencies.
- The combination of frequencies yielding the highest score indicates which digit was dialed in that dialing period.
- MATLAB code follows





## MATLAB code for Decoding Scores

```
pause

%% Decisions
% in each row of the score matrix find the biggest entry among the fir
% four and final three columns
for nn=1:Nnumbers
    [ smax, imax_low(nn)] = max( score(nn, 1:4) );
    [ smax, imax_high(nn)] = max( score(nn, 5:7) );
end

% decode
% lookup table to translate numbers string into numbers
Digits = double('123456789*0#');      % table of ASCII codes for dial-
```

# Part VII

## Frequency Domain Transforms





Lecture: Discrete-Time Fourier Transform

## Introduction

- We will take a closer look at transforming signals into the frequency domain.
  - **Discrete-Time Fourier Transform (DTFT):** applies to arbitrarily long signals; continuous in discrete frequency $f_d$.
  - **z-Transform:** Generalization of DTFT; basis is a complex variable $z$ instead of $e^{j2\pi f_d}$.
  - **Discrete-Fourier Transform:** applies to finite-length signals; computed for discrete set of frequencies; fast algorithms.
- Transforms are useful because:
  - They provide perspectives on signals and systems that aid in signal analysis (e.g., bandwidth)
  - They simplify many problems that are difficult in the time-domain, especially convolution.





## Recall: Frequency Response

- Passing a complex exponential signal $x[n] = \exp(j2\pi f_d n)$ through a linear, time-invariant system with impulse ersponse $h[n]$ yields the output signal

$$y[n] = H(e^{j2\pi f_d}) \cdot \exp(j2\pi f_d n).$$

- The frequency response $H(e^{j2\pi f_d})$ is given by:

$$H(e^{j2\pi f_d}) = \sum_{k=0}^{M-1} h[k] \cdot \exp(-j2\pi f_d k)$$

## Discrete-Time Fourier Transform

- Analogously, we can define for a signal $x[n]$

$$X(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} x[k] \cdot \exp(-j2\pi f_d k)$$

- $X(e^{j2\pi f_d})$ is the Discrete-Time Fourier Transform (DTFT) of the signal $x[n]$; we write

$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}).$$

  - Note that the limits of the sum range from $-\infty$ to $\infty$.
  - To ensure that this infinite sum has a finite value, we must require that

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty.$$





## Two Quick Observations

- **Linearity:** The DTFT is a linear operation.
  - Assume that

$$x_1[n] \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d})$$

    and that

$$x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} X_2(e^{j2\pi f_d}).$$

  - Then,

$$x_1[n] + x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d}) + X_2(e^{j2\pi f_d})$$

- **Periodicity:** The DTFT is periodic in the variable $f_d$:

$$X(e^{j2\pi f_d}) = X(e^{j2\pi (f_d+n)}) \quad \text{for any integer } n.$$

## Continuous-Time Fourier Transform

- In ECE 220, you will learn that the (continuous-time) Fourier transform for a signal $x(t)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot \exp(-j2\pi ft)dt$$

- Notice the strong similarity between the contrinuous and discrete-time transforms.





## DTFT of Delayed Impulse

- Let $x[n]$ be a delayed impulse, $x[n] = \delta[n - n_0]$.
  - Note that $x[n]$ has a single non-zero sample at $n = n_0$.
- Therefore,

$$\begin{aligned} X(e^{j2\pi f_d}) &= \sum_{k=-\infty}^{\infty} x[k] \cdot \exp(-j2\pi f_d k) \\ &= \exp(-j2\pi f_d n_o) \end{aligned}$$

- In summary,

$$\delta[n - n_0] \stackrel{\text{DTFT}}{\longleftrightarrow} \exp(-j2\pi f_d n_o).$$

## DTFT of a Finite-Duration Signal

- Combining Linearity and the DTFT for a delayed impulse, we can easily find the DTFT of a signalk with finitely many samples.

$$\sum_{k=0}^{M-1} x[k] \cdot \delta[n-k] \stackrel{\text{DTFT}}{\longleftrightarrow} \sum_{k=0}^{M-1} x[k] \cdot \exp(-j2\pi f_d k).$$

- Example: The DTFT of the signal $x[n] = \{1, 2, 3, 4\}$ is

$$1 + 2e^{j2\pi f_d} + 3e^{j4\pi f_d} + 4e^{j6\pi f_d}.$$

- I.e.,

$$\{1, 2, 3, 4\} \stackrel{\text{DTFT}}{\longleftrightarrow} 1 + 2e^{j2\pi f_d} + 3e^{j4\pi f_d} + 4e^{j6\pi f_d}$$





## DTFT of a Rectangular Pulse

- Let $x[n]$ be a rectangular pulse of $L$ samples, i.e., $x[n] = u[n] - u[n-L]$.
- Then, the DTFT of $x[n]$ is given by

$$X(e^{j2\pi f_d}) = \sum_{k=0}^{L-1} 1 \cdot e^{j2\pi f_d k}.$$

- Using the *geometric sum formula*

$$S = \sum_{k=0}^{L-1} a^k = \frac{1-a^L}{1-a},$$

$$X(e^{j2\pi f_d}) = \frac{1-e^{-j2\pi f_d L}}{1-e^{-j2\pi f_d}} = \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}.$$

- Thus,

$$\stackrel{\text{DTFT}}{\longleftrightarrow} \sin(\pi f_d L) \quad e^{-j\pi f_d (L-1)}$$

## DTFT of a Right-sided Exponential

- Let $x[n] = a^n \cdot u[n]$ with $|a| < 1$.
- Then, the DTFT of $x[n]$ is given by

$$X(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} a^k \cdot u[k] \cdot e^{-j2\pi f_d k} = \sum_{k=0}^{\infty} a^k \cdot e^{-j2\pi f_d k}.$$

- With the geometric sum formula, we find

$$X(e^{j2\pi f_d}) = \frac{1}{1 - ae^{-j2\pi f_d}}$$

- Thus, if $|a| < 1$

$$a^n \cdot u[n] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j2\pi f_d}}$$





## Inverse DTFT

- The inverse DTFT is used to find the signal $x[n]$ that corresponds to a given transform $X(e^{j2\pi f_d})$.
- The inverse DTFT is given by

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f_d}) e^{j2\pi f_d n} df_d.$$

  - Note: The DTFT is unique, i.e., for each signal $x[n]$ there is exactly one transform $X(e^{j2\pi f_d})$ and vice versa.
  - Explicitly using the inverse transform can often be avoided; instead known DTFT pairs and properties of the DTFT are used; some examples follow.

## Inverse DTFT of $e^{-j2\pi f_d n_0}$

- We showed that the following is a DTFT pair

$$\delta[n-n_0] \xleftrightarrow{\text{DTFT}} \exp(-j2\pi f_d n_o).$$

- Thus, the inverse DTFT of $\exp(-j2\pi f_d n_o)$ must be $\delta[n-n_0]$. Check:
  - For $n = n_0$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-j2\pi f_d n_o) e^{j2\pi f_d n} df_d = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 df_d = 1.$$

  - For $n \neq n_0$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-j2\pi f_d n_o) e^{j2\pi f_d n} df_d = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f_d (n-n_0)} df_d = 0.$$





## Bandlimited Signals

- The inverse DTFT is useful to find signals that are strictly bandlimited.
  - A signal is strictly bandlimited to bandwidth $f_b < \frac{1}{2}$ when its DTFT is given by

$$X(e^{j2\pi f_d}) = \begin{cases} 1 & \text{for } |f_d| \leq f_b \\ 0 & \text{for } f_b < |f_d| \leq \frac{1}{2} \end{cases}$$

- The strictly bandlimited signal is then

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f_d}) e^{j2\pi f_d n} df_d = \frac{\sin(2\pi f_b n)}{\pi n} = 2f_b \cdot \text{sinc}(2\pi f_b n).$$

## Table of DTFT Pairs

$$\delta[n] \stackrel{\text{DTFT}}{\longleftrightarrow} 1$$

$$\delta[n-n_0] \stackrel{\text{DTFT}}{\longleftrightarrow} \exp(-j2\pi f_d n_o)$$

$$u[n]-u[n-L] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}$$

$$a^n \cdot u[n] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1}{1-ae^{-j2\pi f_d}}$$

$$2f_b \cdot \text{sinc}(2\pi f_b n) \stackrel{\text{DTFT}}{\longleftrightarrow} \begin{cases} 1 & \text{for } |f_d| \leq f_b \\ 0 & \text{for } f_b < |f_d| \leq \frac{1}{2} \end{cases}$$





## Exercise

- Find the DTFT of the signals
  1. $$x_1[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3].$$
     - Answer: $X(e^{j2\pi f_d}) = 1 - e^{-j2\pi f_d} + e^{-j4\pi f_d} - e^{-j6\pi f_d}$.
  2. $$x_2[n] = \frac{\sin(2\pi n/4)}{\pi n} + \left(\frac{1}{2}\right)^n \cdot u[n]$$
  3. $$x_3[n] = \left(\frac{1}{2}\right)^n \cdot \cos(2\pi n/3) \cdot u[n]$$

# Lecture: Properties of the DTFT





## Properties of the DTFT

- Direct evaluation of the DTFT or the inverse DTFT is often tedious.
- In many cases, transforms can be determined through a combination of
  - Known, tabulated transform pairs
  - Properties of the DTFT
- Properties of the DTFT describe what happens to the transform when common operations are applied in the time domain (e.g., delay, multiplication with a complex exponential, etc.)
- Very important, a property exists for convolution.

## Linearity

- **Linearity:** The DTFT is a linear operation.
  - Assume that
    $$x_1[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d})$$
    and that
    $$x_2[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_2(e^{j2\pi f_d}).$$
  - Then,

$$x_1[n] + x_2[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d}) + X_2(e^{j2\pi f_d})$$





## Example

- The DTFT of

$$x[n] = \left(\frac{1}{2}\right)^n \cdot u[n] + \frac{\sin(2\pi n/4)}{\pi n}$$

is the sum of the transforms of the two individual signals:

$$X(e^{j2\pi f_d}) = \frac{1}{1 - \frac{1}{2}e^{-j2\pi f_d}} + \begin{cases} 1 & \text{for } |f_d| \leq \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < |f_d| \leq \frac{1}{2} \end{cases}$$

## Time Delay

- Let
$$x[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}).$$
- Find the DTFT of $y[n] = x[n-n_d]$:
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi f_d n} = \sum_{n=-\infty}^{\infty} x[n-n_d] \cdot e^{-j2\pi f_d n}$$
- Substituting $m = n - n_d$ and, therefore, $n = m + n_d$ yields
$$Y(e^{j2\pi f_d}) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f_d (m+n_d)} = e^{-j2\pi f_d n_n} \cdot X(e^{j2\pi f_d})$$
- Hence, the Time Delay property is:
$$x[n-n_d] \stackrel{\text{DTFT}}{\longleftrightarrow} e^{-j2\pi f_d n_n} \cdot X(e^{j2\pi f_d})$$





## Example

- Find the DTFT of a shifted rectangular pulse from 1 to $L+1$
$$x[n] = u[n-1] - u[n-(L+1)].$$
- Combining the DTFT of a rectangular pulse
$$u[n] - u[n-L] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}$$

  with the time delay property leads to
$$u[n-1] - u[n-(L+1)] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L+1)}$$

## Frequency Shift Property

- Let
$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}).$$
- Find the DTFT of $y[n] = x[n] \cdot e^{j2\pi f_0 n}$:
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi f_d n} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f_0 n} \cdot e^{-j2\pi f_d n}$$
- Combining the exponentials yields
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi (f_d - f_0) n} = X(e^{j2\pi (f_d - f_0)})$$
- Frequency shift property
$$x[n] \cdot e^{j2\pi f_0 n} \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi (f_d - f_0)})$$





## Example

- The impulse response of an ideal bandpass filter with bandwidth $B$ and center frequency $f_c$ is obtained by
  - frequency shifting by $f_c$
  - an ideal lowpass with cutoff frequency $B/2$
- Using the transform for the ideal lowpass
$$2f_b \cdot \text{sinc}(2\pi f_b n) \overset{\text{DTFT}}{\longleftrightarrow} \begin{cases} 1 & \text{for } |f_d| \leq f_b \\ 0 & \text{for } f_b < |f_d| \leq \frac{1}{2} \end{cases}$$
the inverse DTFT of the ideal band pass is given by
$$x[n] = B \cdot \text{sinc}(2\pi \frac{B}{2} n) \cdot e^{j2\pi f_c n}$$
  - This technique is very useful to convert lowpass filters into bandpass or highpass filters.

## Convolution Property

- The convolution property follows from linearity and the time delay property.
- Recall that the convolution of signals $x[n]$ and $h[n]$ is defined as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k].$$

- With the time-delay property and linearity, the right hand side transforms to

$$Y(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi f_d k} X(e^{j2\pi f_d}).$$

- Since $\sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi f_d k} = H(e^{j2\pi f_d})$,

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(e^{j2\pi f_d}) \cdot H(e^{j2\pi f_d})$$





## Example

- Convolution of two right sided exponentials ($|a|, |b| < 1$ and $a \neq b$)

$$y[n] = (a^n \cdot u[n]) * (b^n \cdot u[n])$$

has DTFT

$$Y(e^{j2\pi f_d}) = \frac{1}{1 - ae^{-j2\pi f_d}} \cdot \frac{1}{1 - be^{-j2\pi f_d}}$$

- Question: What is the inverse transform of $Y(e^{j2\pi f_d})$? I.e., is there a closed form expression for $y[n]$?

## Example continued

- The expression

$$Y(e^{j2\pi f_d}) = \frac{1}{1 - ae^{-j2\pi f_d}} \cdot \frac{1}{1 - be^{-j2\pi f_d}}$$

can be rewritten as

$$Y(e^{j2\pi f_d}) = \frac{a}{a-b} \cdot \frac{1}{1 - ae^{-j2\pi f_d}} - \frac{b}{a-b} \cdot \frac{1}{1 - be^{-j2\pi f_d}}$$

- The inverse transform of $Y(e^{j2\pi f_d})$ is

$$y[n] = \frac{a}{a-b} \cdot a^n \cdot u[n] - \frac{b}{a-b} \cdot b^n \cdot u[n].$$





## Parseval's Theorem

- The Energy of a discrete-time signal $x[n]$ is defined as

$$E = \sum_{k=-\infty}^{\infty} |x[n]|^2.$$

- Parseval's theorem states that the energy can also be computed using the DTFT

$$E = \sum_{k=-\infty}^{\infty} |x[n]|^= \int_{-\frac{1}{2}}^{\frac{1}{2}} |X(e^{j2\pi f_d})|^2 df_d$$

## Example

- Find the energy of the sinc pulse

$$x[n] = 2f_b \cdot \text{sinc}(2\pi f_b n).$$

- This is impossible in the time domain and trivial in the frequency domain

$$E = \sum_{k=-\infty}^{\infty} |x[n]|^= \int_{-\frac{1}{2}}^{\frac{1}{2}} |X(e^{j2\pi f_d})|^2 df_d = 2f_b$$

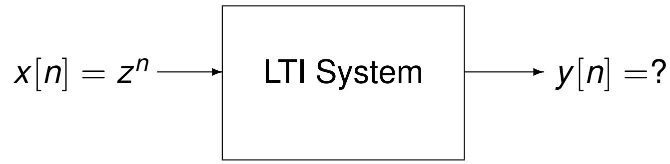


DTFT z-Transform DFT

# Lecture: The $z$-Transform

## Introduction

- **Question:** What is the output of an LTI system when the input is an exponential signal $x[n] = z^n$?
  - $z$ is complex-valued.




- **Answer:**

$$y[n] = H(z) \cdot z^n \quad \text{with} \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] \cdot z^{-n}$$

  - $H(z)$ is the $z$-Transform of the LTI system with impulse response $h[n]$.





## Definitions and Observations

- Analogously, we can define the $z$-Transform of a signal $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

- **Notation:**

$$x[n] \stackrel{z}{\leftrightarrow} X(z).$$

- **Note:** we can think of the ztransform as a generalization of the DTFT.
  - The DTFT arises when $z = e^{j2\pi f_d}$.
- The $z$-Transform is a *linear* operation.

## Examples

- The $z$-Transforms of the following signals generalize easily from the DTFTs computed earlier.

$$\delta[n] \stackrel{z}{\longleftrightarrow} 1$$
$$\delta[n-n_0] \stackrel{z}{\longleftrightarrow} z^{-n_o}$$
$$u[n]-u[n-L] \stackrel{z}{\longleftrightarrow} \frac{1-z^{-L}}{1-z^{-1}}$$
$$a^n \cdot u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$





## $z$-Transform of a Finite Duration Signal

- The $z$-Transform of a signal with finitely many samples is easily computed

$$\sum_{k=0}^{M-1} x[k] \cdot \delta[n-k] \stackrel{z}{\longleftrightarrow} \sum_{k=0}^{M-1} x[k] \cdot z^{-k}.$$

- Example: The DTFT of the signal $x[n] = \{1,2,3,4\}$ is

$$\{1,2,3,4\} \stackrel{z}{\longleftrightarrow} 1+2z^{-1}+3z^{-2}+4z^{-3}$$

- The $z$ transform of a finite-duration signal is a polynomial in $z^{-1}$.
  - The coefficients of the polynomial are the samples of the signal.
  - The inverse $z$-transform is trivial to determine when it is given as a polynomial.

## Properties of the $z$-Transform

Linearity

$$x_1[n] + x_2[n] \overset{z}{\longleftrightarrow} X_z(z) + X_2(z)$$

Delay

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} \cdot X(z)$$

Convolution

$$x[n] * h[n] \overset{z}{\longleftrightarrow} X(z) \cdot H(z)$$





## Unit Delay System

- The unit delay system is an LTI system

$$y[n] = x[n-1]$$

  - Its impulse response and $z$-Transform are is

$$h[n] = \delta[n-1] \quad H(z) = z^{-1}$$

- In terms of the $z$-transform:

$$Y(z) = z^{-1} \cdot X(z)$$

  - In the $z$-domain, a unit delay corresponds to multiplication by $z^{-1}$.
  - In block diagrams, delays are often labeled $z^{-1}$.

## Equivalence of Convolution and Polynomial Multiplcation

- The convolution property states

$$x[n] * h[n] \overset{z}{\longleftrightarrow} X(z) \cdot H(z).$$

- We saw that the $z$-Transforms of finite duration signals are polynomials. Hence, convolution is equivalent to polynomial multiplaction.
- **Example:** $x[n] = \{1, 2, 1\}$ and $h[n] = \{1, 1\}$; by convolution

$$x[n] * h[n] = \{1, 3, 3, , 1\}.$$

- In terms of $z$-Transforms:

$$\begin{aligned} X(z) \cdot H(z) &= (1 + 2z^{-1} + 1z^{-2}) \cdot (1 + 1z^{-1}) \\ &= 1 + 3z^{-1} + 3z^{-2} + z^{-3} \end{aligned}$$





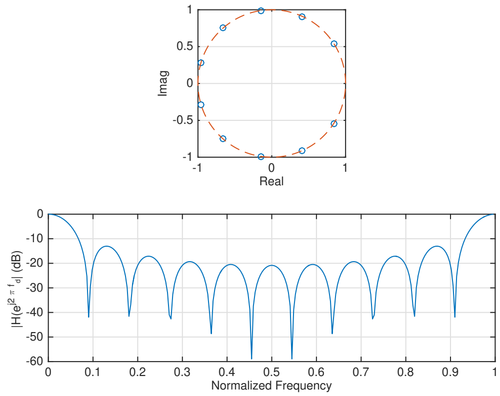
## Zeros of $H(z)$

- An important use of the $z$-Transform is providing insight into the properties of a filter.
- Of particular interest are the zeros of a filter's $z$-Transform $H(z)$.
- **Example:** The $L$-point averager has the $z$-Transform

$$H(z) = \frac{1}{L} \cdot \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{1}{L} \cdot \prod_{k=1}^{L-1} (1 - e^{-j2\pi k/L} \cdot z^{-k}).$$

- The factorization shows that zeros of $H(z)$ occur when $z = e^{-j2\pi k/L}$.
- Note that
  - zeros occur along the unit circle $|z| = 1$
  - at angles that correspond to frequencies $f_d = k/L$ for $k = 1, \ldots, L-1$.
- Zeros are evenly spaced in the stop-band of the filter.

## Roots of $H(z)$ for $L$-Point Averager




Roots of $H(z)$ and magnitude of Frequency Response for $L = 11$-point Averager.





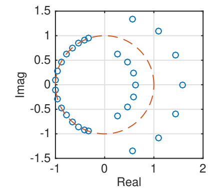
## Roots of $H(z)$ for a very good Lowpass Filter

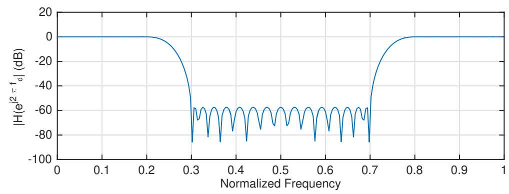
- A very-good lowpass filter with
  - normalized cutoff frequency $f_c = 0.2$ (end of pass passband)
  - width of transition band $\Delta f = 0.1$ (stop band starts at $f_c + \delta f$).

  can be designed in MATLAB with:

```
%% parameters
L = 30;
fc = 0.2; % cutoff frequency - relative to Nyquist frequency
df = 0.1; % width of transition band

%% generate impulse response
h = firpm(L, [0, fc, fc+df, 0.5]/0.5, [1, 1, 0, 0]);
```

## Roots of $H(z)$ for a very good Lowpass Filter




Roots of $H(z)$ and magnitude of Frequency Response for a very good LPF. Zeros are on the unit-circle in the stop band. In the pass band, pairs of roots form a "channel" to keep the frequency response constant.





## IIR Filter

- **Question:** Can we realize a filter with the infinite impulse response (IIR) $h[n] = a^n \cdot u[n]$?
- Recall that
$$a^n \cdot u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}}$$
- Hence,
$$Y(z) = X(Z) \cdot \frac{1}{1 - az^{-1}} \quad \text{or} \quad Y(z) \cdot (1 - az^{-1}) = X(z).$$
- In the time domain,
$$y[n] - ay[n-1] = x[n] \quad \text{or} \quad y[n] = x[n] + ay[n-1].$$

Lecture: Discrete Fourier Transform (DFT)





## Introduction

- The Discrete Fourier Transform (DFT) is a work horse of Digital Signal Processing.
- Its primary uses include:
  - Measuring the spectrum of a signal from samples
  - Fast algorithms for convolution or correlation
- The DFT is computed from a block of $N$ samples $x[0], \ldots, x[N-1]$.
- It computes the DTFT at $N$ evenly spaced, discrete frequencies:

$$X[k] = X(e^{j2\pi \cdot k/N \cdot n}) \quad \text{for } k = 0, \ldots, N-1$$

- Fast algorithms (Fast Fourier Transform (FFT)) exist to compute the DFT.

## Definitions

- **(Forward) Discrete Fourier transform:** for a block of $N$ samples $x[n]$, the DFT $X[k]$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot \exp(-j2\pi \cdot k/N \cdot n) \quad \text{for } k = 0, \ldots, N-1$$

- **Inverse Discrete Fourier transform:** a block of $N$ samples $x[n]$, is obtained from the DFT $X[k]$ by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot \exp(j2\pi \cdot k/N \cdot n) \quad \text{for } n = 0, \ldots, N-1$$





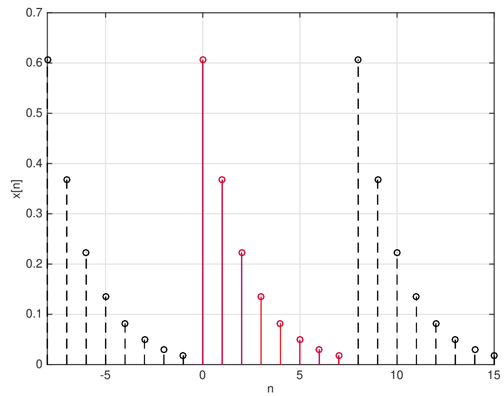
## Observations

- The DFT is *discrete* in **both** time and frequency.
  - In contrast, the DTFT is discrete in time but continuous in frequency.
- The signal $x[n]$ is implicitly assumed to repeat periodically with period $N$.

$$\begin{aligned} x[n+N] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot \exp(j2\pi \cdot k/N \cdot (n+N)) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot \exp(j2\pi \cdot k/N \cdot n) \cdot \exp(j2\pi \cdot k) = x[n] \end{aligned}$$

  - This observation has ramifications for the delay and convolution properties of the DFT.

## Implicit Periodicity




The signal with DFT $X[k]$ is implicitly periodic; the period equals the block length $N$.



DTFT $z$-Transform DFT

## Example

- The DFT[1] of the length $N = 4$ signal $\{1, 1, 0, 0\}$:

$$\begin{aligned} X[0] &= 1e^{-j0} + 1e^{-j0} + 0e^{-j0} + 0e^{-j0} \\ &= 1 + 1 + 0 + 0 = 2 \\ X[1] &= 1e^{-j0} + 1e^{-j2\pi/4} + 0e^{-j4\pi/4} + 0e^{-j6\pi/4} \\ &= 1 + (-j) + 0 + 0 = \sqrt{2}e^{-j\pi/4} \\ X[2] &= 1e^{-j0} + 1e^{-j4\pi/4} + 0e^{-j8\pi/4} + 0e^{-j12\pi/4} \\ &= 1 + (-1) + 0 + 0 = 0 \\ X[3] &= 1e^{-j0} + 1e^{-j6\pi/4} + 0e^{-j12\pi/4} + 0e^{-j18\pi/4} \\ &= 1 + (j) + 0 + 0 = \sqrt{2}e^{j\pi/4} \end{aligned}$$

Thus, $X[k] = \{2, \sqrt{2}e^{-j\pi/4}, 0, \sqrt{2}e^{j\pi/4}\}$

[1] Exponentials are $e^{-j2kn\pi/N}$

## Fast Transform (FFT)

- The main practical benefit of the DFT stems from the fact that a computationally efficient algorithm exists.
- A naive (brute-force) implementation of the DFT requires $N^2$ complex multioplications and additions.
  - $N$ outputs must be computed
  - Each requires $N$ multiplications and additions
- The Fast Fourier Transform algorithm (FFT) reduces the number of complex multiplications and additions to $N \cdot \log_2(N)$.
  - It recursively splits the DFT of length $N$ into 2 DFTs of length $N/2$ (divide-and-conquer)
  - Until length-2 DFTs can be computed trivially.
- A naive DFT of length $N = 1024$ requires approximately $10^6$ multiplications and additions; the FFT requires only approximately $10^4$.





## DFT of a Shifted Impulse

- The finite, length $N$ duration of the signal block and the associated, implicit assumption that $x[n]$ is periodic with period $N$ has some unexpected consequences.
- We showed that the DTFT of a shifted impulse is

$$\delta[n - n_d] \overset{\text{DTFT}}{\longleftrightarrow} e^{-j2\pi f_d n_d}$$

- **DFT with shift $n_d < N$:** assume $N = 8$ and $n_d = 3$

$$X[k] = e^{-j2\pi k/N n_d} = e^{-j3\pi/4k}$$

- **DFT with shift $n_d \geq N$:** assume $N = 8$ and $n_d = 11$

$$X[k] = e^{-j2\pi k/N n_d} = e^{-j11\pi/4k} = e^{-j3\pi/4k} \cdot e{-}j2\pi = e^{-j3\pi/4k}$$

  - Delays induce phase shifts proportional to $n_d$ mod $N$:

$$X[k] = e^{-j2\pi k/N n_d} = e^{-j2\pi k/N(n_d \text{ mod } N)}$$

## Delay Property

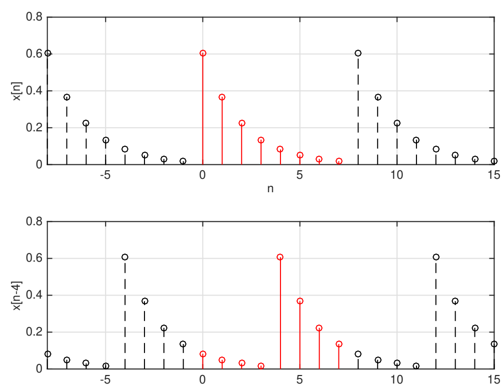
- The same phenomenon affects the delay property.
  - When the implicitly periodic signal is delayed, the block of $N$ samples is filled with periodic samples.
  - For example, when the signal $x[n] = \{1, 2, 3, 4\}$ is shifted by $n_d = 2$ positions it becomes $x[(n - n_d) \bmod N] = \{3, 4, 1, 2\}$.
  - This is refered to as circular shifting.
- For the DFT, the delay property is therefore

$$x[(n - n_d) \bmod N] \overset{\text{DFT}}{\longleftrightarrow} X[k] \cdot e^{-j2\pi k/N n_d}$$





## Implicit Periodicity




Shifting the implicitly periodic signal induces a circular shift over the block of $N$ samples.

## Convolution Property

- Similarly, the convolution property for the DFT is different from that for the DTFT or $z$-Transform.
- A modified form of convolution, called circular convolution has a product-form transform.
  - Let $x[n]$ and $h[n]$ be length-$N$ signals with DFT $X[k]$ and $H[k]$, respectively.
  - Then, the (circular) convolution property is

$$\sum_{m=0}^{N-1} h[m]x[(n-m) \bmod N] \stackrel{\text{DFT}}{\longleftrightarrow} X[k] \cdot H[k]$$

  - Note that circular convolution is very different from normal convolution.
- **Question:** How can the (circular) convolution property be used for fast convolution?



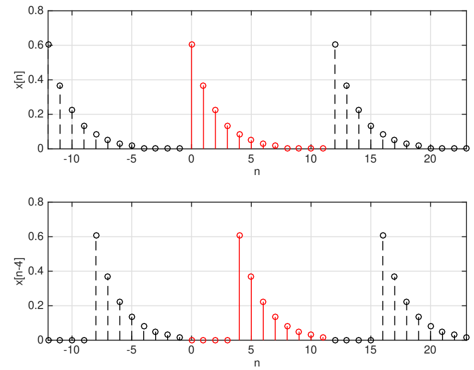


## Zero-Padding

- Turning circular convolution into regular convolution is straightforward:
  - The signals $x[n]$ and $h[n]$ to be convolved must be extended by appending zeros such that
    - They have the same length $N$, and
    - if $x[n]$ has length $N_x$ and $h[n]$ has length $N_h$, then $N \geq N_x + N_h - 1$.
  - This is called zero-padding.
- **Example:** Let $x[n] = \{1, 2, 3, 4\}$ and $h[n] = \{3, 2, 1\}$, then the zero-padded signals are

$$\tilde{x}[n] = \{1, 2, 3, 4, 0, 0\} \quad \tilde{x}[n] = \{3, 2, 1, 0, 0, 0\}$$

## Implicit Periodicity




With zero-padding, the shifting of the implicitly periodic signal introduces only zero samples in the block of $N$ samples.





## Convolution with FFTs

- Fast convolution based on FFTs of zero-padded signals can be implemented as follows:

```
% signals
x = [1,2,3];
h = [1,1];

% zero-padding to length 4
xp = [x, 0];
hp = [h, 0, 0];

% transforms
Xp = fft(xp);
Hp = fft(hp);

% multiply and inverse transform
y = ifft(Xp.*Hp)
```

# Part IX

## Review of Complex Algebra





Lecture: Introduction to Complex Numbers

## Why Complex Numbers?

- Complex numbers are closely related to sinusoids.
- They eliminate the need for trigonometry ...
- ... and replace it with simple algebra.
  - Complex algebra is really simple - this is not an oxymoron.
- Complex numbers can be represented as vectors.
  - Used to visualize the relationship between sinusoids.





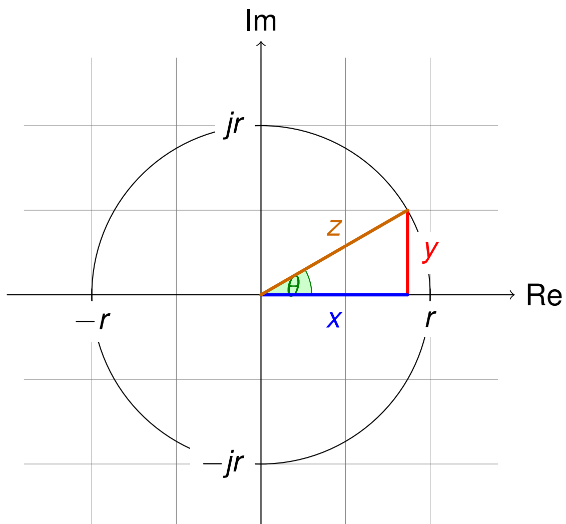
## The Basics

- Complex unity: $j = \sqrt{-1}$.
- Complex numbers can be written as

$$z = x + j \cdot y.$$

  This is called the rectangular or cartesian form.
  - $x$ is called the real part of $z$: $x = \text{Re}\{z\}$.
  - $y$ is called the imaginary part of $z$: $y = \text{Im}\{z\}$.
- $z$ can be thought of a vector in a two-dimensional plane.
  - Cordinates are $x$ and $y$.
  - Coordinate system is called the complex plane.

## Illustration - The Complex Plane








## Euler's Formulas

- Euler's formula provides the connection between complex numbers and trigonometric functions.

$$e^{j\phi} = \cos(\phi) + j \cdot \sin(\phi).$$

- Euler's formula allows conversion between trigonometric functions and exponentials.
  - Exponentials have simple algebraic rules!
- Inverse Euler's formulas:

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

- These relationships are very important.

## Polar Form

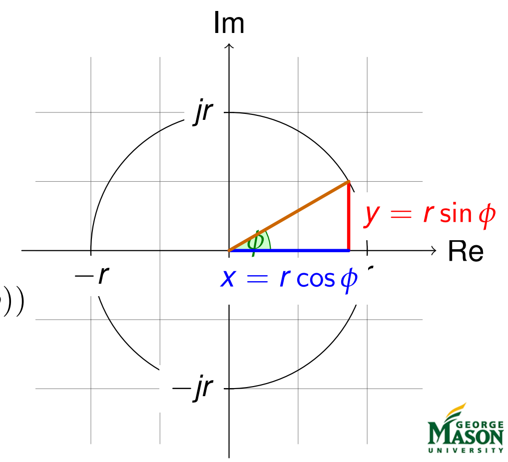
- Recall $z = x + j \cdot y$
- From the diagram it follows that

$$z = r\cos(\phi) + jr\sin(\phi).$$

- And by Euler's relationship:

$$\begin{aligned} z &= r \cdot (\cos(\phi) + j\sin(\phi)) \\ &= r \cdot e^{j\phi} \end{aligned}$$

- This is called the polar form.








## Converting from Polar to Cartesian Form

- Some problems are best solved in rectangular coordinates, while others are easier in polar form.
  - Need to convert between the two forms.
- A complex number polar form $z = r \cdot e^{j\phi}$ is easily converted to cartesian form.

$$z = r\cos(\phi) + jr\sin(\phi).$$

- **Example:**

$$\begin{aligned} 4 \cdot e^{j\pi/3} &= 4 \cdot \cos(\pi/3) + j \cdot 4 \cdot \sin(\pi/3) \\ &= 4 \cdot \tfrac{1}{2} + j \cdot 4 \cdot \tfrac{\sqrt{3}}{2} \\ &= 2 + j \cdot 2 \cdot \sqrt{3}. \end{aligned}$$

## Converting from Cartesian to Polar Form

- A complex number $z = x + jy$ in cartesian form is converted to polar form via

$$r = \sqrt{x^2 + y^2}$$

and

$$\tan(\phi) = \frac{y}{x}.$$

- The computation of the angle $\phi$ requires some care.
- One must distinguish between the cases $x < 0$ and $x > 0$.

$$\phi = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \end{cases}$$

- If $x = 0$, $\phi$ equals $+\pi/2$ or $-\pi/2$ depending on the sign of $y$.





## Exercise

- Convert to polar form
  1. $z = 1 + j$
  2. $z = 3 \cdot j$
  3. $z = -1 - j$
- Convert to cartesian form
  1. $z = 3e^{-j3\pi/4}$
- in MATLAB, plot $cos(jx)$ for $-2 \leq x \leq 2$ then explain the shape of the resulting graph.

## Lecture: Complex Algebra





## Introduction

- All *normal* rules of algebra apply to complex numbers!
- One thing to look for: $j \cdot j = -1$.
- Some operations are best carried out in rectangular coordinates.
  - Addition and subtraction
  - Multiplication and division aren't very hard, either.
- Others are easier in polar coordinates.
  - Multiplication and division.
  - Powers and roots
- New operation: conjugate complex.
- A little more subtle: absolute value.

## Conjugate Complex

- The *conjugate complex* $z^*$ of a complex number $z$ has
  - the same real part as $z$: $\text{Re}\{z\} = \text{Re}\{z^*\}$, and
  - the opposite imaginary part: $\text{Im}\{z\} = -\text{Im}\{z^*\}$.
- **Rectangular form:**

$$\text{If } z = x + jy \text{ then } z^* = x - jy.$$

- **Polar form:**

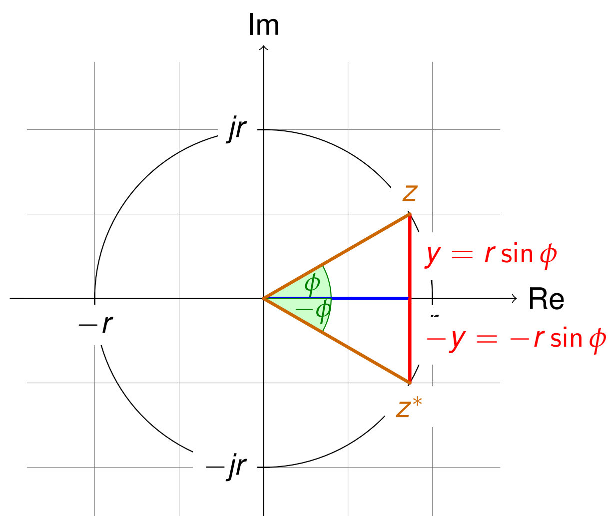
$$\text{If } z = r \cdot e^{j\phi} \text{ then } z^* = r \cdot e^{-j\phi}.$$

- Note, $z$ and $z^*$ are mirror images of each other in the complex plane with respect to the real axis.





## Illustration - Conjugate Complex



## Addition and Subtraction

- Addition and subtraction can only be done in rectangular form.
  - If the complex numbers to be added are in polar form convert to rectangular form, first.
- Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$.
- **Addition:**

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- **Subtraction:**

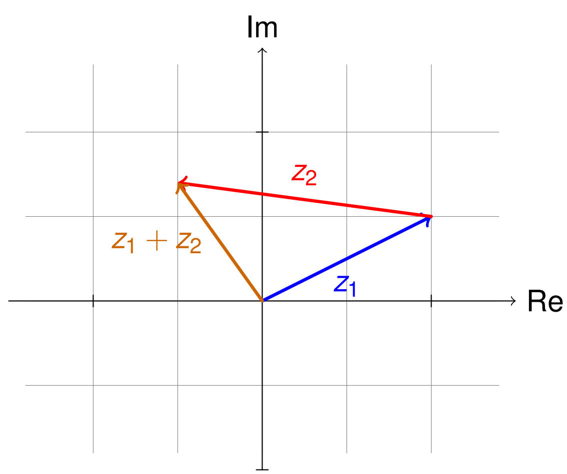
$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- Complex addition works like *vector addition.*





## Illustration - Complex Addition



## Multiplication

- Multiplication of complex numbers is possible in both polar and rectangular form.
- **Polar Form:** Let $z_1 = r_1 \cdot e^{j\phi_1}$ and $z_2 = r_2 \cdot e^{j\phi_2}$, then

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \exp(j(\phi_1 + \phi_2)).$$

- **Rectangular Form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + jy_1) \cdot (x_2 + jy_2) \\ &= x_1 x_2 + j^2 y_1 y_2 + j x_1 y_2 + j x_2 y_1 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

- Polar form provides more insight: multiplication involves rotation in the complex plane (because of $\phi_1 + \phi_2$).





## Absolute Value

- The absolute value of a complex number $z$ is defined as

$$|z| = \sqrt{z \cdot z^*}, \text{ thus, } |z|^2 = z \cdot z^*.$$

  - Note, $|z|$ and $|z|^2$ are real-valued.
  - In MATLAB, `abs(z)` computes $|z|$.
- **Polar Form:** Let $z = r \cdot e^{j\phi}$,

$$|z|^2 = r \cdot e^{j\phi} \cdot r \cdot e^{-j\phi} = r^2.$$

  - Hence, $|z| = r$.
- **Rectangular Form:** Let $z = x + jy$,

$$\begin{aligned} |z|^2 &= (x + jy) \cdot (x - jy) \\ &= x^2 - j^2 y^2 - jxy + jxy \\ &= x^2 + y^2. \end{aligned}$$

## Division

- Closely related to multiplication

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}.$$

- **Polar Form:** Let $z_1 = r_1 \cdot e^{j\phi_1}$ and $z_2 = r_2 \cdot e^{j\phi_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \exp(j(\phi_1 - \phi_2)).$$

- **Rectangular Form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{z_1 z_2^*}{|z_2|^2} \\ &= \frac{(x_1+jy_1)\cdot(x_2-jy_2)}{x_2^2+y_2^2} \\ &= \frac{(x_1x_2+y_1y_2)+j(-x_1y_2+x_2y_1)}{x_2^2+y_2^2}. \end{aligned}$$





## Exercises

- For $z_1 = 3e^{j\pi/4}$ and $z_2 = 2e^{-j\pi/2}$, compute
  1. $z_1 + z_2$,
  2. $z_1 \cdot z_2$, and
  3. $|z_1|$.

  Give your results in both polar and rectangular forms.

## Lecture: Complex Algebra - Continued





## Good to know ...

- You should try and remember the following relationships and properties.
  - $e^{j2\pi} = 1$
  - $e^{j\pi} = -1$
  - $e^{j\pi/2} = j$
  - $e^{-j\pi/2} = -j$
  - $|e^{j\phi}| = 1$ for all values of $\phi$
  - $\exp(j(\phi + 2\pi)) = e^{j\phi}$

## Powers of Complex Numbers

- A complex number $z$ is easily raised to the $n$-th power if $z$ is in polar form.
- Specifically,

$$\begin{aligned} z^n &= (r \cdot e^{j\phi})^n \\ &= r^n \cdot e^{jn\phi} \end{aligned}$$

  - The magnitude $r$ is raised to the $n$-th power
  - The phase $\phi$ is multiplied by $n$.
- The above holds for arbitrary values of $n$, including
  - $n$ an integer (e.g., $z^2$),
  - $n$ a fraction (e.g., $z^{1/2} = \sqrt{z}$)
  - $n$ a negative number (e.g., $z^{-1} = 1/z$)
  - $n$ a complex number (e.g., $z^j$)





## Roots of Unity

- Quite often all complex numbers $z$ solving the following equation must be found

$$z^N = 1.$$

  - Here $N$ is an integer.
  - There are $N$ different complex numbers solving this equation.
- The solutions have the form

$$z = e^{j2\pi n/N} \text{ for } n = 0, 1, 2, \ldots, N-1.$$

  - Note that $z^N = e^{j2\pi n} = 1$!
- The solutions are called the $N$-th roots of unity.
- In the complex plane, all solutions lie on the unit circle and are separated by angle $2\pi/N$.

## Roots of a Complex Number

- The more general problem is to find *all* solutions of the equation
$$z^N = r \cdot e^{j\phi}.$$
- In this case, the $N$ solutions are given by
$$z = r^{1/N} \cdot \exp(j\frac{\phi + 2\pi n}{N}) \text{ for } n = 0, 1, 2, \ldots, N-1.$$

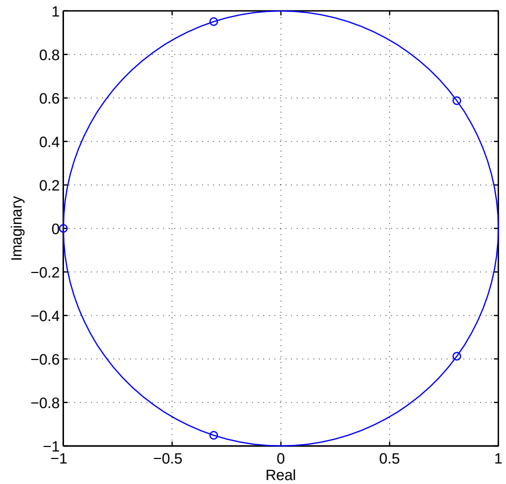




## Example: Roots of a Complex Number

- **Example:** Find all solutions of $z^5 = -1$.
- **Solution:**
  - Note $-1 = e^{j\pi}$, i.e., $r = 1$ and $\phi = \pi$.
  - There are $N = 5$ solutions:
    - All have magnitude 1.
    - The five angles are $\pi/5$, $3\pi/5$, $5\pi/5$, $7\pi/5$, $9\pi/5$.

## Roots of a Complex Number








## Two Ways to Express $\cos(\phi)$

- First relationship: $\cos(\phi) = \text{Re}\{e^{j\phi}\}$
- Second relationship (inverse Euler):

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}.$$

- The first form is best suited as the starting point for problems involving the cosine or sine of a sum.
  - $\cos(\alpha + \beta)$
- The second form is best when products of sines and cosines are needed
  - $\cos(\alpha) \cdot \cos(\beta)$
- Rule of thumb: look to create products of exponentials.

## Example

- Show that $\cos(x+y)$ equals $\cos(x)\cos(y) - \sin(x)\sin(y)$:

$$\begin{aligned}\cos(x+y) &= \mathrm{Re}\{e^{j(x+y)}\} = \mathrm{Re}\{e^{jx}\cdot e^{jy}\}\\ &= \mathrm{Re}\{(\cos(x)+j\sin(x))\cdot(\cos(y)+j\sin(y))\}\\ &= \mathrm{Re}\{(\cos(x)\cos(y)-\sin(x)\sin(y))+\\ &\quad j(\cos(x)\sin(y)+\sin(x)\cos(y))\}\\ &= \cos(x)\cos(y)-\sin(x)\sin(y).\end{aligned}$$





## Example

- Show that $\cos(x)\cos(y)$ equals $\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$:

$$\begin{aligned}\cos(x)\cos(y) &= \frac{e^{jx}+e^{-jx}}{2}\,\frac{e^{jy}+e^{-jy}}{2}\\ &= \frac{e^{j(x+y)}+e^{j(-x-y)}+e^{j(x-y)}+e^{j(-x+y)}}{4}\\ &= \frac{e^{j(x+y)}+e^{-j(x+y)}}{4}+\frac{e^{j(x-y)}+e^{-j(x-y)}}{4}\\ &= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y).\end{aligned}$$

# Exercises

- Simplify
  1. $(\sqrt{2} - \sqrt{2}j)^8$
  2. $(\sqrt{2} - \sqrt{2}j)^{-1}$
- Advanced
  1. $j^j$
  2. $\cos(j)$