Part I

Spectrum Representation of Signals
Lecture: Sums of Sinusoids (of different frequency)
To this point we have focused on sinusoids of identical frequency $f$

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i).$$

- Note that the frequency $f$ does not have a subscript $i$!
- Showed (in phasor addition rule) that the above sum can always be written as a single sinusoid of frequency $f$. 
We will consider sums of sinusoids of different frequencies:

\[ x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

- Note the subscript on the frequencies \( f_i \)!
- This apparently minor difference has dramatic consequences.
Sum of Two Sinusoids

\[ x(t) = \frac{4}{\pi} \cos(2\pi ft - \pi/2) + \frac{4}{3\pi} \cos(2\pi 3ft - \pi/2) \]
Sum of 25 Sinusoids

\[ x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusoids in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids of frequencies \((2n - 1) \cdot f\), and amplitudes \(4 \frac{1}{(2n-1)\pi}\).
  - (This is not obvious).
The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.

Basic idea: list the “ingredients” of a signal by specifying amplitudes and phases as well as frequencies of the sinusoids in the sum.
The Spectrum of a Sum of Sinusoids

- Begin with the sum of sinusoids introduced earlier

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

where we have broken out a possible constant term.

- The term \( A_0 \) can be thought of as corresponding to a sinusoid of frequency zero.

- Using the *inverse Euler formula*, we can replace the sinusoids by complex exponentials

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i}. \)
The Spectrum of a Sum of Sinusoids (cont’d)

Starting with

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i} \).

The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

\[ X(f) = \{(X_0, 0), \left(\frac{1}{2}X_1, f_1\right), \left(\frac{1}{2}X_1^*, -f_1\right), \ldots, \left(\frac{1}{2}X_N, f_N\right), \left(\frac{1}{2}X_N^*, -f_N\right)\} \]
Example

Consider the signal

\[ x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4). \]

Using the inverse Euler relationship

\[ x(t) = 3 + \frac{5}{2} e^{-j\pi/2} \exp(j2\pi 10t) + \frac{5}{2} e^{j\pi/2} \exp(-j2\pi 10t) + \frac{7}{2} e^{j\pi/4} \exp(j2\pi 25t) + \frac{7}{2} e^{-j\pi/4} \exp(-j2\pi 25t). \]

Hence,

\[ X(f) = \{(3, 0), \left(\frac{5}{2} e^{-j\pi/2}, 10\right), \left(\frac{5}{2} e^{j\pi/2}, -10\right), \left(\frac{7}{2} e^{j\pi/4}, 25\right), \left(\frac{7}{2} e^{-j\pi/4}, -25\right)\} \]
Exercise

Find the spectrum of the signal:

\[ x(t) = 6 + 4 \cos(10\pi t + \pi/3) + 5 \cos(20\pi t - \pi/7). \]
Lecture: From Time-Domain to Frequency-Domain and back
Signals are *naturally* observed in the time-domain.

A signal can be illustrated in the time-domain by plotting it as a function of time.

The frequency-domain provides an alternative perspective of the signal based on sinusoids:

- Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
- The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
- Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of these parameters.
  - Actually, we list pairs of complex amplitudes ($Ae^{j\phi}$) and frequencies $f$ and refer to this list as $X(f)$. 
It is possible (but not necessarily easy) to find \( X(f) \) from \( x(t) \): this is called Fourier or spectrum \textit{analysis}.

Similarly, one can construct \( x(t) \) from the spectrum \( X(f) \): this is called Fourier \textit{synthesis}.

Notation: \( x(t) \leftrightarrow X(f) \).

Example (from last time):

\begin{itemize}
  \item \textbf{Time-domain:} signal
  \[ x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4). \]
  \item \textbf{Frequency Domain:} spectrum
  \[ X(f) = \{(3, 0), \left(\frac{5}{2} e^{-j\pi/2}, 10\right), \left(\frac{5}{2} e^{j\pi/2}, -10\right), \left(\frac{7}{2} e^{j\pi/4}, 25\right), \left(\frac{7}{2} e^{-j\pi/4}, -25\right)\} \]
\end{itemize}
To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.

Sometimes the phase is plotted versus frequency as well.
In many applications, the frequency contents of a signal is very important.

- For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
- Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.

Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).

We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.

- Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.
Example: Original signal
Example: Corrupted signal
Synthesis is a straightforward process; it is a lot like following a recipe.

*Ingredients* are given by the spectrum

\[ X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\} \]

Each pair indicates one complex exponential component by listing its frequency and complex amplitude.

*Instructions* for combining the ingredients and producing the (time-domain) signal:

\[ x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t). \]

You should simplify the expression you obtain.
Example

- Problem: Find the signal $x(t)$ corresponding to

$$X(f) = \{(3, 0), \left(\frac{5}{2} e^{-j\pi/2}, 10\right), \left(\frac{5}{2} e^{j\pi/2}, -10\right), \left(\frac{7}{2} e^{j\pi/4}, 25\right), \left(\frac{7}{2} e^{-j\pi/4}, -25\right)\}$$

- Solution:

$$x(t) = 3 + \frac{5}{2} e^{-j\pi/2} e^{j2\pi 10t} + \frac{5}{2} e^{j\pi/2} e^{-j2\pi 10t} + \frac{7}{2} e^{j\pi/4} e^{j2\pi 25t} + \frac{7}{2} e^{-j\pi/4} e^{-j2\pi 25t}$$

- Which simplifies to:

$$x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4).$$
Exercise

Find the signal with the spectrum:

\[ X(f) = \{(5, 0), (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \\
\quad (\frac{5}{2}e^{j\pi/4}, 15), (\frac{5}{2}e^{-j\pi/4}, -15) \} \]
The objective of spectrum or Fourier analysis is to find the spectrum of a time-domain signal.

We will restrict ourselves to signals $x(t)$ that are sums of sinusoids

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

We have already shown that such signals have spectrum:

$$X(f) = \{(X_0, 0), (\frac{1}{2} X_1, f_1), (\frac{1}{2} X_1^*, -f_1), \ldots, (\frac{1}{2} X_N, f_N), (\frac{1}{2} X_N^*, -f_N)\}$$

where $X_0 = A_0$ and $X_i = A_i e^{i\phi_i}$.

We will investigate some interesting signals that can be written as a sum of sinusoids.
Consider the signal

\[ x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t). \]

This signal does not have the form of a sum of sinusoids; hence, we can not determine its spectrum immediately.
% BeatNote — plot and play a beat note waveform

% Parameters
fs = 8192;
dur = 2;
NP = round(fs/5);

f1 = 5;
f2 = 400;
A = 2;

% time axis
tt = 0:1/fs:dur;

xx = A*cos(2*pi*f1*tt).*cos(2*pi*f2*tt);

plot(tt(1:NP), xx(1:NP))
xlabel('Time (s)')
soundsc(xx, fs);
Beat Notes as a Sum of Sinusoids

Using the inverse Euler relationships, we can write

\[ x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t) \]
\[ = 2 \cdot \frac{1}{2} \cdot (e^{j2\pi 5t} + e^{-j2\pi 5t}) \cdot \frac{1}{2} \cdot (e^{j2\pi 400t} + e^{-j2\pi 400t}). \]

Multiplying out yields:

\[ x(t) = \frac{1}{2} (e^{j2\pi 405t} + e^{-j2\pi 405t}) + \frac{1}{2} (e^{j2\pi 395t} + e^{-j2\pi 395t}). \]

Applying Euler’s relationship, lets us write:

\[ x(t) = \cos(2\pi 405t) + \cos(2\pi 395t). \]
Spectrum of Beat Notes

- We were able to rewrite the beat notes as a sum of sinusoids

\[ x(t) = \cos(2\pi 405t) + \cos(2\pi 395t). \]

- Note that the frequencies in the sum, 395 Hz and 405 Hz, are the sum and difference of the frequencies in the original product, 5 Hz and 400 Hz.

- It is now straightforward to determine the spectrum of the beat notes signal:

\[ X(f) = \left\{ \left( \frac{1}{2}, 405 \right), \left( \frac{1}{2}, -405 \right), \left( \frac{1}{2}, 395 \right), \left( \frac{1}{2}, -395 \right) \right\} \]
Spectrum of Beat Notes
Lecture: Amplitude Modulation and Periodic Signals
Amplitude Modulation

- **Amplitude Modulation** is used in communication systems.
- The objective of amplitude modulation is to move the spectrum of a signal $m(t)$ from low frequencies to high frequencies.
  - The message signal $m(t)$ may be a piece of music; its spectrum occupies frequencies below 20 KHz.
  - For transmission by an AM radio station this spectrum must be moved to approximately 1 MHz.
Amplitude Modulation

- Conventional amplitude modulation proceeds in two steps:
  - A constant \( A \) is added to \( m(t) \) such that \( A + m(t) > 0 \) for all \( t \).
  - The sum signal \( A + m(t) \) is multiplied by a sinusoid \( \cos(2\pi f_c t) \), where \( f_c \) is the radio frequency assigned to the station.

- Consequently, the transmitted signal has the form:

\[
x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).
\]
We are interested in the spectrum of the AM signal.

However, we cannot compute $X(f)$ for arbitrary message signals $m(t)$.

For the special case $m(t) = \cos(2\pi f_m t)$ we can find the spectrum.

- To mimic the radio case, $f_m$ would be a frequency in the audible range.

As before, we will first need to express the AM signal $x(t)$ as a sum of sinusoids.
Amplitude Modulated Signal

- For \( m(t) = \cos(2\pi f_m t) \), the AM signal equals
  \[
x(t) = (A + \cos(2\pi f_m t)) \cdot \cos(2\pi f_c t).
  \]

- This simplifies to
  \[
x(t) = A \cdot \cos(2\pi f_c t) + \cos(2\pi f_m t) \cdot \cos(2\pi f_c t).
  \]

- Note that the second term of the sum is a beat notes signal with frequencies \( f_m \) and \( f_c \).

- We know that beat notes can be written as a sum of sinusoids with frequencies equal to the sum and difference of \( f_m \) and \( f_c \):
  \[
x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2} \cos(2\pi (f_c + f_m) t) + \frac{1}{2} \cos(2\pi (f_c - f_m) t).
  \]
The AM signal is given by

\[ x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2} \cos(2\pi (f_c + f_m) t) + \frac{1}{2} \cos(2\pi (f_c - f_m) t). \]

Thus, its spectrum is

\[ X(f) = \{ (\frac{A}{2}, f_c), (\frac{A}{2}, -f_c), \]
\[ (\frac{1}{4}, f_c + f_m), (\frac{1}{4}, -f_c - f_m), (\frac{1}{4}, f_c - f_m), (\frac{1}{2}, -f_c + f_m) \} \]
For $A = 2$, $fm = 50$, and $fc = 400$, the spectrum of the AM signal is plotted below.
Spectrum of Amplitude Modulated Signal

- It is interesting to compare the spectrum of the signal before modulation and after multiplication with \( \cos(2\pi f_c t) \).
- The signal \( s(t) = A + m(t) \) has spectrum
  \[
  S(f) = \{(A, 0), \left(\frac{1}{2}, 50\right), \left(\frac{1}{2}, -50\right)\}.
  \]
- The modulated signal \( x(t) \) has spectrum
  \[
  X(f) = \{(\frac{A}{2}, 400), (\frac{A}{2}, -400),
  (\frac{1}{4}, 450), (\frac{1}{4}, -450), (\frac{1}{4}, 350), (\frac{1}{2}, -350)\}
  \]
- Both are plotted on the next page.
Sum of Sinusoidal Signals
Time-Domain and Frequency-Domain
Periodic Signals
Time-Frequency Spectrum

Why Bother with the Frequency-Domain?
Synthesis: From Frequency to Time-Domain
Analysis: From Time to Frequency-Domain
Amplitude Modulation

Spectrum before and after AM

![Graph showing spectrum before and after AM modulation.](image)
Comparison of the two spectra shows that amplitude modulation indeed moves a spectrum from low frequencies to high frequencies.

Note that the shape of the spectrum is precisely preserved.

Amplitude modulation can be described concisely by stating:

- Half of the original spectrum is shifted by $f_c$ to the right, and the other half is shifted by $f_c$ to the left.

**Question:** How can you get the original signal back so that you can listen to it.

- This is called demodulation.
What are Periodic Signals?

- A signal \( x(t) \) is called **periodic** if there is a constant \( T_0 \) such that

\[ x(t) = x(t + T_0) \] for all \( t \).

- In other words, a periodic signal repeats itself every \( T_0 \) seconds.

- The interval \( T_0 \) is called the **fundamental period** of the signal.

- The inverse of \( T_0 \) is the **fundamental frequency** of the signal.

- Example:
  - A sinusoidal signal of frequency \( f \) is periodic with period
    \[ T_0 = 1/f. \]
Consider a sum of sinusoids:

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

A special case arises when we constrain all frequencies \( f_i \) to be integer multiples of some frequency \( f_0 \):

\[ f_i = i \cdot f_0. \]

The frequencies \( f_i \) are then called harmonic frequencies of \( f_0 \).

We will show that sums of sinusoids with frequencies that are harmonics are periodic.
To establish periodicity, we must show that there is \( T_0 \) such that 
\[ x(t) = x(t + T_0). \]

Begin with
\[
x(t + T_0) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i (t + T_0) + \phi_i)
\]
\[
= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i)
\]

Now, let \( f_0 = 1/T_0 \) and use the fact that frequencies are harmonics: \( f_i = i \cdot f_0 \).
Harmonic Signals are Periodic

Then, \( f_i \cdot T_0 = i \cdot f_0 \cdot T_0 = i \) and hence

\[
x(t + T_0) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i)
\]
\[
= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi i + \phi_i)
\]

We can drop the \( 2\pi i \) terms and conclude that \( x(t + T_0) = x(t) \).

**Conclusion:** A signal of the form

\[
x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i \cdot f_0 t + \phi_i)
\]

is periodic with period \( T_0 = 1/f_0 \).
Finding the Fundamental Frequency

- Often one is given a set of frequencies $f_1, f_2, \ldots, f_N$ and is required to find the fundamental frequency $f_0$.
- Specifically, this means one must find a frequency $f_0$ and integers $n_1, n_2, \ldots, n_N$ such that all of the following equations are met:

\[
\begin{align*}
    f_1 &= n_1 \cdot f_0 \\
    f_2 &= n_2 \cdot f_0 \\
    &\vdots \\
    f_N &= n_N \cdot f_0
\end{align*}
\]

- Note that there isn’t always a solution to the above problem.
  - However, if all frequencies are integers a solution exists.
  - Even if all frequencies are rational a solution exists.
Example

- Find the fundamental frequency for the set of frequencies $f_1 = 12, f_2 = 27, f_3 = 51$.
- Set up the equations:

$$
12 = n_1 \cdot f_0 \\
27 = n_2 \cdot f_0 \\
51 = n_3 \cdot f_0 
$$

- Try the solution $n_1 = 1$; this would imply $f_0 = 12$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 2$; this would imply $f_0 = 6$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 3$; this would imply $f_0 = 4$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 4$; this would imply $f_0 = 3$. This **can** satisfy the other two equations with $n_2 = 9$ and $n_3 = 17$. 

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Example

Note that the three sinusoids complete a cycle at the same time at $T_0 = 1/f_0 = 1/3\,\text{s}$. 
Exercise

Find the fundamental frequency for the set of frequencies $f_1 = 2, f_2 = 3.5, f_3 = 5$. 
We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.

One can turn this statement around and arrive at a very important result:

Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.

The resulting sum is called the Fourier Series of the signal.

Put differently, a periodic signal can always be written in the form

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi if_0 t + \phi_i) \]
\[ = X_0 + \sum_{i=1}^{N} X_i e^{j2\pi if_0 t} + X_i^* e^{-j2\pi if_0 t} \]

with \( X_0 = A_0 \) and \( X_i = \frac{A_i}{2} e^{j\phi_i} \).
For a periodic signal the complex amplitudes $X_i$ can be computed using a (relatively) simple formula.

Specifically, for a periodic signal $x(t)$ with fundamental period $T_0$ the complex amplitudes $X_i$ are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi i t / T_0} dt.$$

Note that the integral above can be evaluated over any interval of length $T_0$. 
Example: Square Wave

A square wave signal

\[ x(t) = \begin{cases} 
1 & 0 \leq t < \frac{T_0}{2} \\
-1 & \frac{T_0}{2} \leq t < T_0 
\end{cases} \]

can be written as

\[ x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
25-Term Approximation to Square Wave

\[ x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
Lecture: Time-Frequency Spectrum
So far, we have considered only signals that can be written as a sum of sinusoids.

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

For such signals, we are able to compute the spectrum.

Note, that signals of this form
- are assumed to last forever, i.e., for \(-\infty < t < \infty\),
- and their spectrum never changes.

While such signals are important and useful conceptually, they don’t describe real-world signals accurately.

Real-world signals
- are of finite duration,
- their spectrum changes over time.
Musical Notation

- Musical notation ("sheet music") provides a way to represent real-world signals: a piece of music.
- As you know, sheet music
  - places notes on a scale to reflect the frequency of the tone to be played,
  - uses differently shaped note symbols to indicate the duration of each tone,
  - provides the order in which notes are to be played.

In summary, musical notation captures how the spectrum of the music-signal changes over time.

- We cannot write signals whose spectrum changes with time as a sum of sinusoids.
  - A static spectrum is insufficient to describe such signals.
- Alternative: time-frequency spectrum
Example: Musical Scale

<table>
<thead>
<tr>
<th>Note</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>262</td>
<td>294</td>
<td>330</td>
<td>349</td>
<td>392</td>
<td>440</td>
<td>494</td>
<td>523</td>
</tr>
</tbody>
</table>

**Table:** Musical Notes and their Frequencies
Example: Musical Scale

- If we play each of the notes for 250 ms, then the resulting signal can be summarized in the time-frequency spectrum below.
MATLAB has a function `spectrogram` that can be used to compute the time-frequency spectrum for a given signal.

- The resulting plots are similar to the one for the musical scale on the previous slide.

Typically, you invoke this function as `spectrogram(xx, 256, 128, 256, fs)`, where `xx` is the signal to be analyzed and `fs` is the sampling frequency.

The spectrogram for the musical scale is shown on the next slide.
The color indicates the magnitude of the spectrum at a given time and frequency.
Objective: construct a signal such that its frequency increases with time.

Starting Point: A sinusoidal signal has the form:
\[ x(t) = A \cos(2\pi f_0 t + \phi). \]

We can consider the argument of the cos as a time-varying phase function
\[ \Psi(t) = 2\pi f_0 t + \phi. \]

Question: What happens when we allow more general functions for \( \Psi(t) \)?
  - For example, let
\[ \Psi(t) = 700\pi t^2 + 440\pi t + \phi. \]
Spectrogram: $\cos(\psi(t))$

**Question:** How is the time-frequency spectrum related to $\psi(t)$?
For a regular sinusoid, \( \Psi(t) = 2\pi f_0 t + \phi \) and the frequency equals \( f_0 \).

This suggests as a possible relationship between \( \Psi(t) \) and \( f_0 \):

\[
f_0 = \frac{1}{2\pi} \frac{d}{dt} \Psi(t).
\]

If the above derivative is not a constant, it is called the instantaneous frequency of the signal, \( f_i(t) \).

**Example:** For \( \Psi(t) = 700\pi t^2 + 440\pi t + \phi \) we find

\[
f_i(t) = \frac{1}{2\pi} \frac{d}{dt} (700\pi t^2 + 440\pi t + \phi) = 700t + 220.
\]

This describes precisely the red line in the spectrogram on the previous slide.
Constructing a Linear Chirp

- **Objective:** Construct a signal such that its frequency is initially $f_1$ and increases linear to $f_2$ after $T$ seconds.

- **Solution:** The above suggests that

$$f_i(t) = \frac{f_2 - f_1}{T} t + f_1.$$  

Consequently, the phase function $\Psi(t)$ must be

$$\Psi(t) = 2\pi \frac{f_2 - f_1}{2T} t^2 + 2\pi f_1 t + \phi$$

Note that $\phi$ has no influence on the spectrum; it is usually set to 0.
Constructing a Linear Chirp

**Example:** Construct a linear chirp such that the frequency decreases from 1000 Hz to 200 Hz in 2 seconds.

The desired signal must be

\[ x(t) = \cos(-2\pi 200t^2 + 2\pi 1000t). \]
Exercise

- Construct a linear chirp such that the frequency increases from 50 Hz to 200 Hz in 3 seconds.
- Sketch the time-frequency spectrum of the following signal

\[ x(t) = \cos(2\pi 500t + 100 \cos(2\pi 2t)) \]