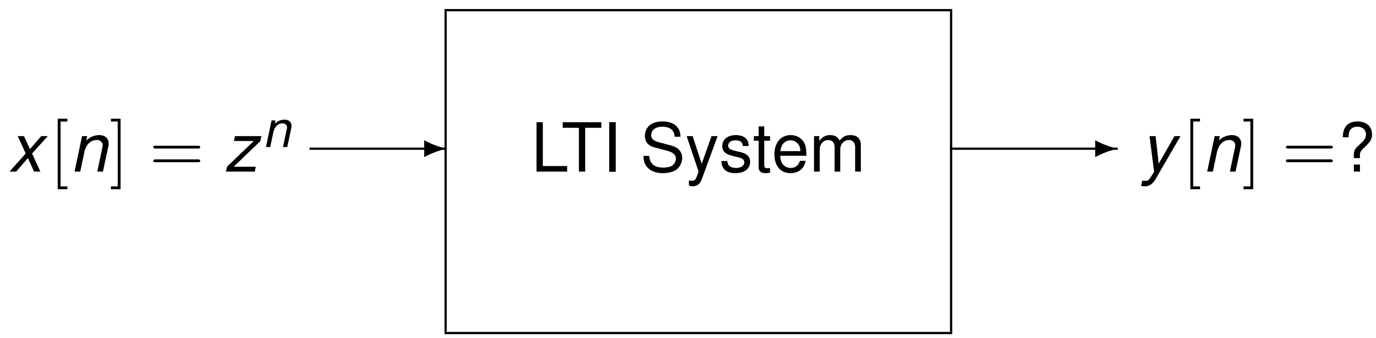
Introduction

- **Question:** What is the output of an LTI system when the input is an exponential signal $x[n] = z^n$?
  - z is complex-valued.



- **Answer:**

$$y[n] = H(z) \cdot z^n \quad \text{with} \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] \cdot z^{-n}$$

  - $H(z)$ is the z-Transform of the LTI system with impulse response $h[n]$.

Definitions and Observations

- Analogously, we can define the z-Transform of a signal $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

- **Notation:**

$$x[n] \stackrel{z}{\longleftrightarrow} X(z).$$

- **Note:** we can think of the $z$transform as a generalization of the DTFT.
  - The DTFT arises when $z = e^{j2\pi f_d}$.
- The z-Transform is a *linear* operation.

Examples

- The z-Transforms of the following signals generalize easily from the DTFTs computed earlier.

$$\delta[n] \stackrel{z}{\longleftrightarrow} 1$$

$$\delta[n-n_0] \stackrel{z}{\longleftrightarrow} z^{-n_o}$$

$$u[n]-u[n-L] \stackrel{z}{\longleftrightarrow} \frac{1-z^{-L}}{1-z^{-1}}$$

$$a^n \cdot u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$

z-Transform of a Finite Duration Signal

- The z-Transform of a signal with finitely many samples is easily computed

$$\sum_{k=0}^{M-1} x[k] \cdot \delta[n-k] \stackrel{z}{\longleftrightarrow} \sum_{k=0}^{M-1} x[k] \cdot z^{-k}.$$

- Example: The DTFT of the signal $x[n] = \{1, 2, 3, 4\}$ is

$$\{1, 2, 3, 4\} \stackrel{z}{\longleftrightarrow} 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

- The z transform of a finite-duration signal is a polynomial in z^{-1}.
  - The coefficients of the polynomial are the samples of the signal.
  - The inverse z-transform is trivial to determine when it is given as a polynomial.

Properties of the z-Transform

Linearity

$$x_1[n] + x_2[n] \stackrel{z}{\longleftrightarrow} X_z(z) + X_2(z)$$

Delay

$$x[n - n_0] \stackrel{z}{\longleftrightarrow} z^{-n_0} \cdot X(z)$$

Convolution

$$x[n] * h[n] \stackrel{z}{\longleftrightarrow} X(z) \cdot H(z)$$

Unit Delay System

- The unit delay system is an LTI system

$$y[n] = x[n-1]$$

  - Its impulse response and z-Transform are is

$$h[n] = \delta[n-1] \quad H(z) = z^{-1}$$

- In terms of the z-transform:

$$Y(z) = z^{-1} \cdot X(z)$$

  - In the z-domain, a unit delay corresponds to multiplication by z^{-1}.
  - In block diagrams, delays are often labeled z^{-1}.

Equivalence of Convolution and Polynomial Multiplcation

- The convolution property states

$$x[n] * h[n] \overset{z}{\longleftrightarrow} X(z) \cdot H(z).$$

- We saw that the z-Transforms of finite duration signals are polynomials. Hence, convolution is equivalent to polynomial multiplaction.
- **Example:** $x[n] = \{1, 2, 1\}$ and $h[n] = \{1, 1\}$; by convolution

$$x[n] * h[n] = \{1, 3, 3, , 1\}.$$

- In terms of z-Transforms:

$$\begin{aligned} X(z) \cdot H(z) &= (1 + 2z^{-1} + 1z^{-2}) \cdot (1 + 1z^{-1}) \\ &= 1 + 3z^{-1} + 3z^{-2} + z^{-3} \end{aligned}$$

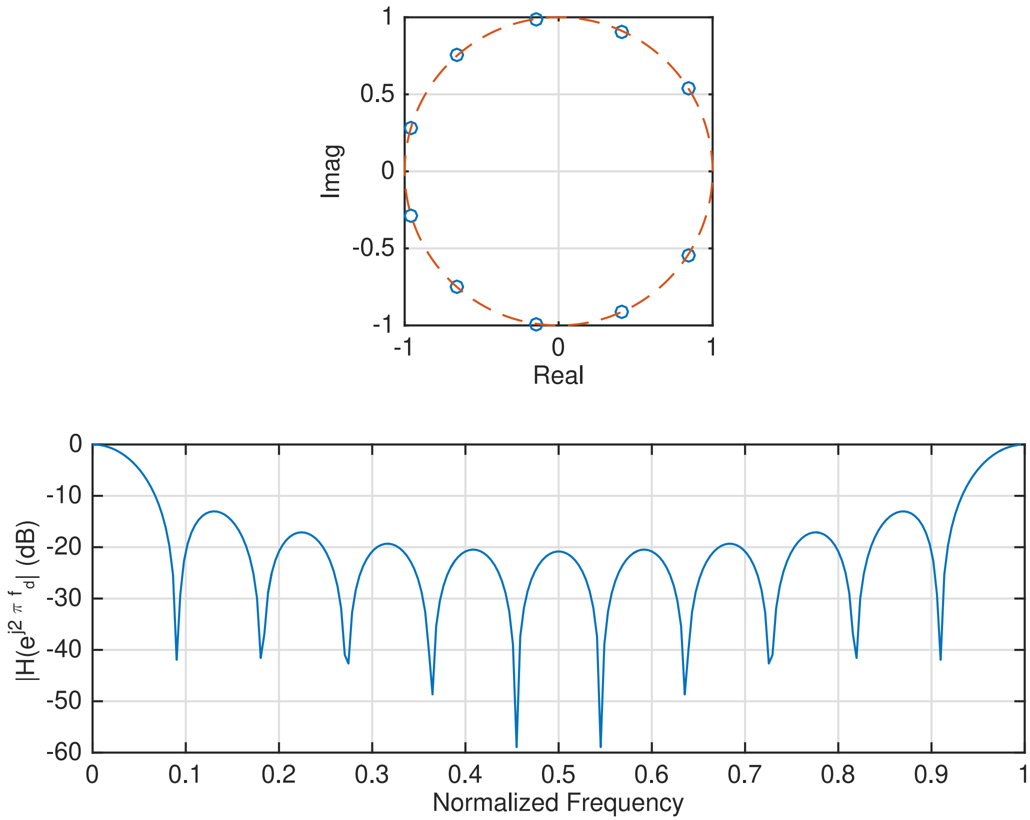
Zeros of $H(z)$

- An important use of the z-Transform is providing insight into the properties of a filter.
- Of particular interest are the zeros of a filter's z-Transform $H(z)$.
- **Example:** The L-point averager has the z-Transform

$$H(z) = \frac{1}{L} \cdot \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{1}{L} \cdot \prod_{k=1}^{L-1} (1 - e^{-j2\pi k/L} \cdot z^{-k}).$$

- The factorization shows that zeros of $H(z)$ occur when $z = e^{-j2\pi k/L}$.
- Note that
  - zeros occur along the unit circle $|z| = 1$
  - at angles that correspond to frequencies $f_d = k/L$ for $k = 1, \ldots, L-1$.
- Zeros are evenly spaced in the stop-band of the filter.

Roots of $H(z)$ for L-Point Averager



Roots of $H(z)$ and magnitude of Frequency Response for $L = 11$-point Averager.

Roots of $H(z)$ for a very good Lowpass Filter

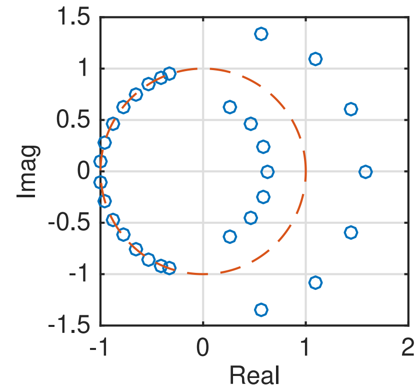
- A very-good lowpass filter with
  - normalized cutoff frequency $f_c = 0.2$ (end of pass passband)
  - width of transition band $\Delta f = 0.1$ (stop band starts at $f_c + \delta f$).

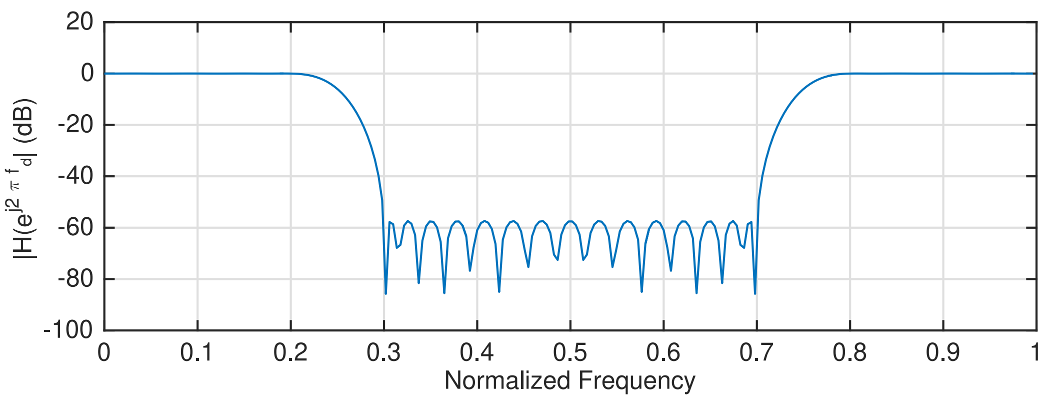
  can be designed in MATLAB with:

```
%% parameters
L = 30;
fc = 0.2; % cutoff frequency - relative to Nyquist frequency
df = 0.1; % width of transition band

%% generate impulse response
h = firpm(L, [0, fc, fc+df, 0.5]/0.5, [1, 1, 0, 0]);
```

Roots of $H(z)$ for a very good Lowpass Filter





Roots of $H(z)$ and magnitude of Frequency Response for a very good LPF. Zeros are on the unit-circle in the stop band. In the pass band, pairs of roots form a “channel” to keep the frequency response constant.

IIR Filter

- **Question:** Can we realize a filter with the infinite impulse response (IIR) $h[n] = a^n \cdot u[n]$?
- Recall that

$$a^n \cdot u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

- Hence,

$$Y(z) = X(Z) \cdot \frac{1}{1 - az^{-1}} \quad \text{or} \quad Y(z) \cdot (1 - az^{-1}) = X(z).$$

- In the time domain,

$$y[n] - ay[n-1] = x[n] \quad \text{or} \quad y[n] = x[n] + ay[n-1].$$