Lecture: Properties of the DTFT

Properties of the DTFT

- Direct evaluation of the DTFT or the inverse DTFT is often tedious.
- In many cases, transforms can be determined through a combination of
  - Known, tabulated transform pairs
  - Properties of the DTFT
- Properties of the DTFT describe what happens to the transform when common operations are applied in the time domain (e.g., delay, multiplication with a complex exponential, etc.)
- Very important, a property exists for convolution.

Linearity

- **Linearity:** The DTFT is a linear operation.
  - Assume that

$$x_1[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d})$$

    and that

$$x_2[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_2(e^{j2\pi f_d}).$$

  - Then,

$$x_1[n] + x_2[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X_1(e^{j2\pi f_d}) + X_2(e^{j2\pi f_d})$$

Example

- The DTFT of

$$x[n] = \left(\frac{1}{2}\right)^n \cdot u[n] + \frac{\sin(2\pi n/4)}{\pi n}$$

is the sum of the transforms of the two individual signals:

$$X(e^{j2\pi f_d}) = \frac{1}{1 - \frac{1}{2}e^{-j2\pi f_d}} + \begin{cases} 1 & \text{for } |f_d| \leq \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < |f_d| \leq \frac{1}{2} \end{cases}$$

Time Delay

- Let
$$x[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}).$$
- Find the DTFT of $y[n] = x[n - n_d]$:
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi f_d n} = \sum_{n=-\infty}^{\infty} x[n - n_d] \cdot e^{-j2\pi f_d n}$$
- Substituting $m = n - n_d$ and, therefore, $n = m + n_d$ yields
$$Y(e^{j2\pi f_d}) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f_d (m+n_d)} = e^{-j2\pi f_d n_n} \cdot X(e^{j2\pi f_d})$$
- Hence, the Time Delay property is:
$$x[n - n_d] \stackrel{\text{DTFT}}{\longleftrightarrow} e^{-j2\pi f_d n_n} \cdot X(e^{j2\pi f_d})$$

Example

- Find the DTFT of a shifted rectangular pulse from 1 to $L+1$

$$x[n] = u[n-1] - u[n-(L+1)].$$

- Combining the DTFT of a rectangular pulse

$$u[n] - u[n-L] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}$$

with the time delay property leads to

$$u[n-1] - u[n-(L+1)] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L+1)}$$

Frequency Shift Property

- Let
$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}).$$
- Find the DTFT of $y[n] = x[n] \cdot e^{j2\pi f_0 n}$:
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi f_d n} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f_0 n} \cdot e^{-j2\pi f_d n}$$
- Combining the exponentials yields
$$Y(e^{j2\pi f_d}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j2\pi (f_d - f_0) n} = X(e^{j2\pi (f_d - f_0)})$$
- Frequency shift property
$$x[n] \cdot e^{j2\pi f_0 n} \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi (f_d - f_0)})$$

Example

- The impulse response of an ideal bandpass filter with bandwidth B and center frequency f_c is obtained by
  - frequency shifting by f_c
  - an ideal lowpass with cutoff frequency $B/2$
- Using the transform for the ideal lowpass

$$2f_b \cdot \operatorname{sinc}(2\pi f_b n) \overset{\text{DTFT}}{\longleftrightarrow} \begin{cases} 1 & \text{for } |f_d| \leq f_b \\ 0 & \text{for } f_b < |f_d| \leq \frac{1}{2} \end{cases}$$

the inverse DTFT of the ideal band pass is given by

$$x[n] = B \cdot \operatorname{sinc}(2\pi \frac{B}{2} n) \cdot e^{j2\pi f_c n}$$

  - This technique is very useful to convert lowpass filters into bandpass or highpass filters.

Convolution Property

- The convolution property follows from linearity and the time delay property.
- Recall that the convolution of signals $x[n]$ and $h[n]$ is defined as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k].$$

- With the time-delay property and linearity, the right hand side transforms to

$$Y(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi f_d k} X(e^{j2\pi f_d}).$$

- Since $\sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi f_d k} = H(e^{j2\pi f_d})$,

$$x[n] * h[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(e^{j2\pi f_d}) \cdot H(e^{j2\pi f_d})$$

Example

- Convolution of two right sided exponentials ($|a|, |b| < 1$ and $a \neq b$)

$$y[n] = (a^n \cdot u[n]) * (b^n \cdot u[n])$$

  has DTFT

$$Y(e^{j2\pi f_d}) = \frac{1}{1 - ae^{-j2\pi f_d}} \cdot \frac{1}{1 - be^{-j2\pi f_d}}$$

- Question: What is the inverse transform of $Y(e^{j2\pi f_d})$? I.e., is there a closed form expression for $y[n]$?