# Introduction

- We will take a closer look at transforming signals into the frequency domain.
  - **Discrete-Time Fourier Transform (DTFT):** applies to arbitrarily long signals; continuous in discrete frequency $f_d$.
  - **z-Transform:** Generalization of DTFT; basis is a complex variable $z$ instead of $e^{j2\pi f_d}$.
  - **Discrete-Fourier Transform:** applies to finite-length signals; computed for discrete set of frequencies; fast algorithms.
- Transforms are useful because:
  - They provide perspectives on signals and systems that aid in signal analysis (e.g., bandwidth)
  - They simplify many problems that are difficult in the time-domain, especially convolution.

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# Recall: Frequency Response

- Passing a complex exponential signal $x[n] = \exp(j2\pi f_d n)$ through a linear, time-invariant system with impulse ersponse $h[n]$ yields the output signal

$$y[n] = H(e^{j2\pi f_d}) \cdot \exp(j2\pi f_d n).$$

- The frequency response $H(e^{j2\pi f_d})$ is given by:

$$H(e^{j2\pi f_d}) = \sum_{k=0}^{M-1} h[k] \cdot \exp(-j2\pi f_d k)$$

# Discrete-Time Fourier Transform

- Analogously, we can define for a signal $x[n]$
$$X(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} x[k] \cdot \exp(-j2\pi f_d k)$$
- $X(e^{j2\pi f_d})$ is the Discrete-Time Fourier Transform (DTFT) of the signal $x[n]$; we write
$$x[n] \xleftrightarrow{DTFT} X(e^{j2\pi f_d}).$$
  - Note that the limits of the sum range from $-\infty$ to $\infty$.
  - To ensure that this infinite sum has a finite value, we must require that
$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty.$$

## Two Quick Observations

- **Linearity:** The DTFT is a linear operation.
  - Assume that
  $$x_1[n] \xleftrightarrow{DTFT} X_1(e^{j2\pi f_d})$$
  and that
  $$x_2[n] \xleftrightarrow{DTFT} X_2(e^{j2\pi f_d}).$$
  - Then,
  $$x_1[n] + x_2[n] \xleftrightarrow{DTFT} X_1(e^{j2\pi f_d}) + X_2(e^{j2\pi f_d})$$
- **Periodicity:** The DTFT is periodic in the variable $f_d$:
$$X(e^{j2\pi f_d}) = X(e^{j2\pi (f_d+n)}) \quad \text{for any integer } n.$$

# Continuous-Time Fourier Transform

- In ECE 220, you will learn that the (continuous-time) Fourier transform for a signal $x(t)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot \exp(-j2\pi ft) dt$$

- Notice the strong similarity between the contrinuous and discrete-time transforms.

# DTFT of Delayed Impulse

- Let $x[n]$ be a delayed impulse, $x[n] = \delta[n - n_0]$.
  - Note that $x[n]$ has a single non-zero sample at $n = n_0$.
- Therefore,

$$X(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} x[k] \cdot \exp(-j2\pi f_d k) = \exp(-j2\pi f_d n_o)$$

- In summary,

$$\delta[n - n_0] \xleftrightarrow{DTFT} \exp(-j2\pi f_d n_o).$$

# DTFT of a Finite-Duration Signal

- Combining Linearity and the DTFT for a delayed impulse, we can easily find the DTFT of a signalk with finitely many samples.

$$\sum_{k=0}^{M-1} x[k] \cdot \delta[n-k] \overset{DTFT}{\longleftrightarrow} \sum_{k=0}^{M-1} x[k] \cdot \exp(-j2\pi f_d k).$$

- Example: The DTFT of the signal $x[n] = \{1, 2, 3, 4\}$ is

$$1 + 2e^{j2\pi f_d} + 3e^{j4\pi f_d} + 4e^{j6\pi f_d}.$$

- I.e.,

$$\{1, 2, 3, 4\} \overset{DTFT}{\longleftrightarrow} 1 + 2e^{j2\pi f_d} + 3e^{j4\pi f_d} + 4e^{j6\pi f_d}$$

# DTFT of a Rectangular Pulse

- Let $x[n]$ be a rectangular pulse of $L$ samples, i.e., $x[n] = u[n] - u[n-L]$.
- Then, the DTFT of $x[n]$ is given by

$$X(e^{j2\pi f_d}) = \sum_{k=0}^{L-1} 1 \cdot e^{j2\pi f_d k}.$$

- Using the *geometric sum formula*

$$S = \sum_{k=0}^{L-1} a^k = \frac{1-a^L}{1-a},$$

$$X(e^{j2\pi f_d}) = \frac{1-e^{-j2\pi f_d L}}{1-e^{-j2\pi f_d}} = \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}.$$

- Thus,

$$x[n] = u[n] - u[n-L] \overset{DTFT}{\longleftrightarrow} \frac{\sin(\pi f_d L)}{\sin(\pi f_d)} \cdot e^{-j\pi f_d (L-1)}$$

# DTFT of a Right-sided Exponential

- Let $x[n] = a^n \cdot u[n]$ with $|a| < 1$.
- Then, the DTFT of $x[n]$ is given by

$$X(e^{j2\pi f_d}) = \sum_{k=-\infty}^{\infty} a^k \cdot u[k] \cdot e^{-j2\pi f_d k} = \sum_{k=0}^{\infty} a^k \cdot e^{-j2\pi f_d k}.$$

- With the geometric sum formula, we find

$$X(e^{j2\pi f_d}) = \frac{1}{1 - ae^{-j2\pi f_d}}$$

- Thus, if $|a| < 1$

$$a^n \cdot u[n] \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j2\pi f_d}}$$

# Inverse DTFT

- The inverse DTFT is used to find the signal $x[n]$ that corresponds to a given transform $X(e^{j2\pi f_d})$.
- The inverse DTFT is given by

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f_d}) e^{j2\pi f_d n} df_d.$$

  - Note: The DTFT is unique, i.e., for each signal $x[n]$ there is exactly one transform $X(e^{j2\pi f_d})$ and vice versa.
  - Explicitly using the inverse transform can often be avoided; instead known DTFT pairs and properties of the DTFT are used; some examples follow.

# Inverse DTFT of $e^{-j2\pi f_d n_0}$

- We showed that the following is a DTFT pair

$$\delta[n-n_0] \stackrel{DTFT}{\longleftrightarrow} \exp(-j2\pi f_d n_o).$$

- Thus, the inverse DTFT of $\exp(-j2\pi f_d n_o)$ must be $\delta[n-n_0]$. Check:
  - For $n = n_0$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-j2\pi f_d n_o) e^{j2\pi f_d n} df_d = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \, df_d = 1.$$

  - For $n \neq n_0$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-j2\pi f_d n_o) e^{j2\pi f_d n} df_d = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f_d (n-n_0)} df_d = 0.$$

# Bandlimited Signals

- The inverse DTFT is useful to find signals that are strictly bandlimited.
  - A signal is strictly bandlimited to bandwidth $f_b < \frac{1}{2}$ when its DTFT is given by

$$X(e^{j2\pi f_d}) = \begin{cases} 1 & \text{for } |f_d| \leq f_b \\ 0 & \text{for } f_b < |f_d| \leq \frac{1}{2} \end{cases}$$

- The strictly bandlimited signal is then

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f_d}) e^{j2\pi f_d n} df_d = \frac{\sin(2\pi f_b n)}{\pi n} = 2f_b \cdot \text{sinc}(2\pi f_b n).$$

## Exercise

- Find the DTFT of the signals
  1. $$x_1[n] = 1 - e^{-j2\pi f_d} + e^{-j4\pi f_d} - e^{-j6\pi f_d}.$$
  2. $$x_2[n] = \frac{\sin(2\pi n/4)}{\pi n} + \left(\frac{1}{2}\right)^n \cdot u[n]$$
  3. $$x_3[n] = \left(\frac{1}{2}\right)^n \cdot \cos(2\pi n/3) \cdot u[n]$$