Introduction

- We have demonstrated that for linear, time-invariant systems
  - the output signal $y[n]$
  - is the convolution of the input signal $x[n]$ and the impulse response $h[n]$.

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=0}^{M} h[k] \cdot x[n-k]
\end{aligned}
$$

- **Question:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi fn + \phi))$.

Response to a Complex Exponential

- **Problem:** Find the output signal $y[n]$ when the input signal is $x[n] = A\exp(j(2\pi fn + \phi)$.
- Output $y[n]$ is convolution of input and impulse response

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=0}^{M} h[k] \cdot x[n-k] \\
&= \sum_{k=0}^{M} h[k] \cdot A\exp(j(2\pi f(n-k) + \phi) \\
&= A\exp(j(2\pi fn + \phi) \cdot \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi fk) \\
&= A\exp(j(2\pi fn + \phi) \cdot H(f)
\end{aligned}
$$

- The term

$$
H(f) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi fk)
$$

is called the Frequency Response of the system.

Interpreting the Frequency Response

The Frequency Response of an LTI system with impulse response $h[n]$ is

$$H(f) = \sum_{k=0}^{M} h[k] \cdot \exp(-j2\pi fk)$$

- **Observations:**
  - The response of a LTI system to a complex exponential signal is a complex exponential signal of the same frequency.
    - Complex exponentials are eigenfunctions of LTI systems.
  - When $x[n] = A\exp(j(2\pi fn + \phi))$, then $y[n] = x[n] \cdot H(f)$.
    - This is true only for complex exponential input signals!

Interpreting the Frequency Response

- **Observations:**
  - $H(f)$ is best interpreted in polar coordinates:

$$H(f) = |H(f)| \cdot e^{j\angle H(f)}.$$

    - Then, for $x[n] = A\exp(j(2\pi fn + \phi)$

$$\begin{aligned} y[n] &= x[n] \cdot H(f) \\ &= A\exp(j(2\pi fn + \phi) \cdot |H(f)| \cdot e^{j\angle H(f)} \\ &= (A \cdot |H(f)|) \cdot \exp(j(2\pi fn + \phi + \angle H(f)) \end{aligned}$$

  - The amplitude of the resulting complex exponential is the product $A \cdot |H(f)|$.
    - Therefore, $|H(f)|$ is called the gain of the system.
  - The phase of the resulting complex exponential is the sum $\phi + \angle H(f)$.
    - $\angle H(f)$ is called the phase of the system.

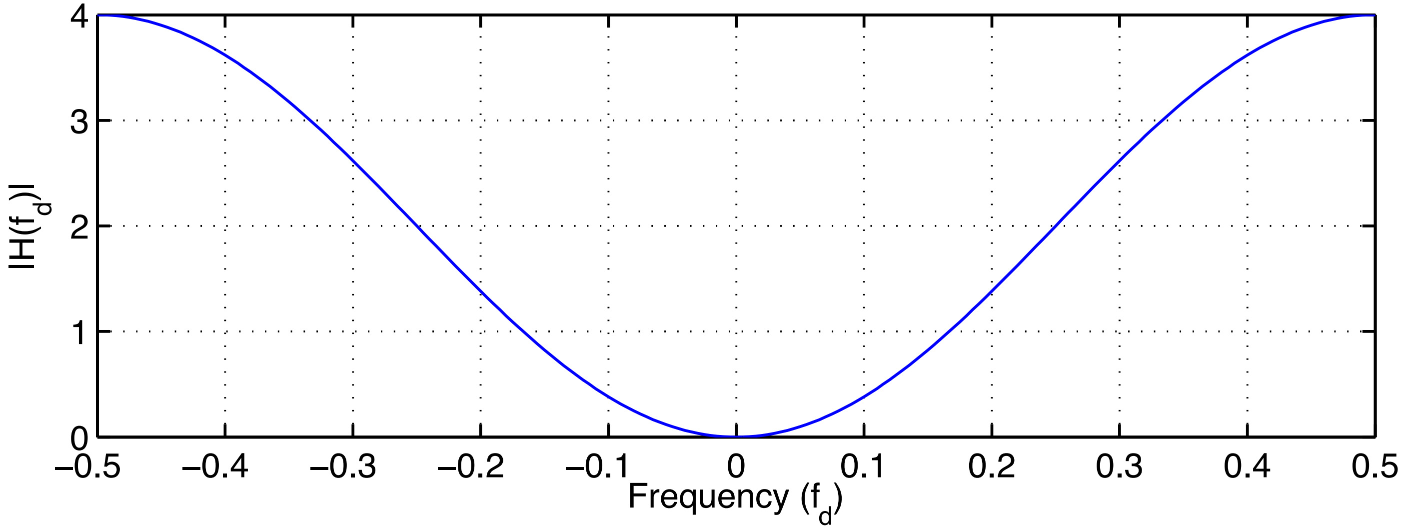
Example

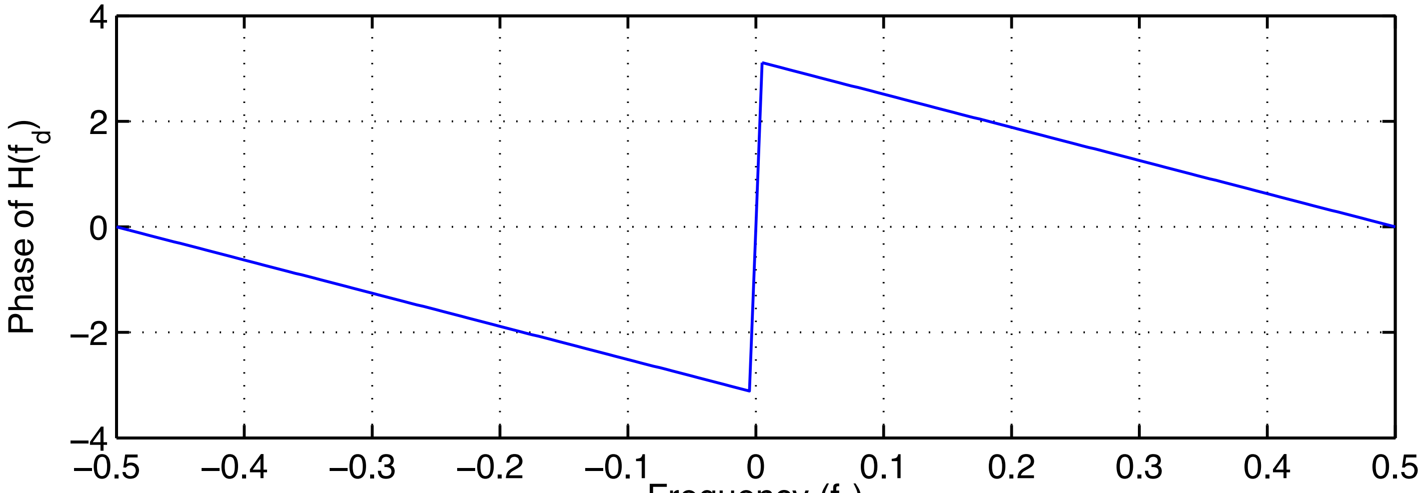
- Let $h[n] = \{1, -2, 1\}$.
- Then,

$$\begin{aligned} H(f) &= \sum_{k=0}^{2} h[k] \cdot \exp(-j2\pi fk) \\ &= 1 - 2 \cdot \exp(-j2\pi f) + 1 \cdot \exp(-j2\pi f2) \\ &= \exp(-j2\pi f) \cdot (\exp(j2\pi f) - 2 + \exp(-j2\pi f)) \\ &= \exp(-j2\pi f) \cdot (2\cos(2\pi f) - 2). \end{aligned}$$

- Gain: $|H(f)| = |(2\cos(2\pi f) - 2|$

Example


4
3
2
1
0
|H(f_d)|
–0.5 –0.4 –0.3 –0.2 –0.1 0 0.1 0.2 0.3 0.4 0.5
Frequency (f_d)


4
2
0
–2
–4
Phase of H(f_d)
–0.5 –0.4 –0.3 –0.2 –0.1 0 0.1 0.2 0.3 0.4 0.5

Example

- The filter with impulse response $h[n] = \{1, -2, 1\}$ is a high-pass filter.
  - It rejects sinusoids with frequencies near $f = 0$,
  - and passes sinusoids with frequencies near $f = \frac{1}{2}$
- Note how the function of this system is much easier to describe in terms of the frequency response $H(f)$ than in terms of the impulse response $h[n]$.
- **Question:** Find the output signal when input equals $x[n] = 2\exp(j2\pi 1/4n - \pi/2)$.
- **Solution:**

$$H(\frac{1}{4}) = \exp(-j2\pi\frac{1}{4}) \cdot (2\cos(2\pi\frac{1}{4}) - 2) = -2e^{-j\pi/2} = 2e^{j\pi/2}.$$

Thus,

$$y[n] = 2e^{j\pi/2} \cdot x[n] = 4\exp(j2\pi n/4).$$

Exercise

1. Find the Frequency Response $H(f)$ for the LTI system with impulse response $h[n] = \{1, -1, -1, 1\}$.
2. Find the output for the input signal $x[n] = 2\exp(j(2\pi n/3 - \pi/4))$.