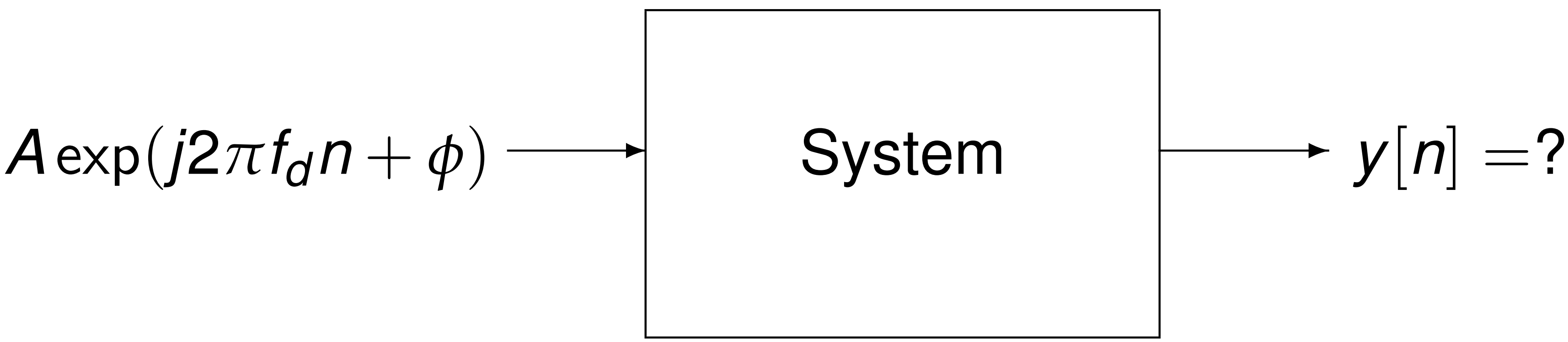
Introduction

- We have discussed:
  - Sinusoidal and complex exponential signals,
  - Spectrum representation of signals:
    - arbitrary signals can be expressed as the sum of sinusoidal (or complex exponential) signals.
  - Linear, time-invariant systems.
- Next: complex exponential signals as input to linear, time-invariant systems.



Example: 3-Point Averaging Filter

- Consider the 3-point averager:

$$y[n] = \frac{1}{3}\sum_{k=0}^{2} x[n-k] = \frac{1}{3}\cdot(x[n] + x[n-1] + x[n-2]).$$

- **Question:** What is the output $y[n]$ if the input is $x[n] = \exp(j2\pi f_d n)$?
  - Recall that f_d is the normalized frequency f/f_s; we are assuming the signal is oversampled, $|f_d| < \frac{1}{2}$
  - Initially, assume $A = 1$ and $\phi = 0$; generalization is easy.

Delayed Complex Exponentials

- The 3-point averager involves delayed versions of the input signal.
- We begin by assessing the impact the delay has on the complex exponential input signal.
- For

$$x[n] = \exp(j2\pi f_d n)$$

a delay by k samples leads to

$$\begin{aligned} x[n-k] &= \exp(j2\pi f_d(n-k)) \\ &= e^{j(2\pi f_d n - 2\pi f_d k)} = e^{j2\pi f_d n} \cdot e^{-j2\pi f_d k} \\ &= e^{j(2\pi f_d n + \phi_k)} = e^{j2\pi f_d n} \cdot e^{j\phi_k} \end{aligned}$$

where $\phi_k = -2\pi f_d k$ is the phase shift induced by the k sample delay.

Average of Delayed Complex Exponentials

- Now, the output signal $y[n]$ is the average of three delayed complex exponentials

$$\begin{aligned} y[n] &= \frac{1}{3}\sum_{k=0}^{2} x[n-k] \\ &= \frac{1}{3}\sum_{k=0}^{2} e^{j(2\pi f_d n - 2\pi f_d k)} \end{aligned}$$

- This expression involves the sum of complex exponentials of the same frequency; the phasor addition rule applies:

$$y[n] = e^{j2\pi f_d n} \cdot \frac{1}{3}\sum_{k=0}^{2} e^{-j2\pi f_d k}.$$

- **Important Observation:** The output signal is a complex exponential of the same frequency as the input signal.
  - The amplitude and phase are different.

Frequency Response of the 3-Point Averager

- The output signal $y[n]$ can be rewritten as:

$$\begin{aligned} y[n] &= e^{j2\pi f_d n} \cdot \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi f_d k} \\ &= e^{j2\pi f_d n} \cdot H(e^{j2\pi f_d}). \end{aligned}$$

where

$$\begin{aligned} H(e^{j2\pi f_d}) &= \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi f_d k} \\ &= \frac{1}{3} \cdot (1 + e^{-j2\pi f_d} + e^{-j2\pi 2 f_d}) \\ &= \frac{1}{3} \cdot e^{-j2\pi f_d} (e^{j2\pi f_d} + 1 + e^{-j2\pi f_d}) \\ &= \frac{e^{-j2\pi f_d}}{3} (1 + 2\cos(2\pi f_d)). \end{aligned}$$

Interpretation

- From the above, we can conclude:
  - If the input signal is of the form $x[n] = \exp(j2\pi f_d n)$,
  - then the output signal is of the form $y[n] = H(e^{j2\pi f_d}) \cdot \exp(j2\pi f_d n)$.
- The function $H(e^{j2\pi f_d})$ is called the frequency response of the system.
- **Note:** If we know $H(e^{j2\pi f_d})$, we can easily compute the output signal in response to a complex expontial input signal.

Examples

- Recall:

$$H(e^{j2\pi f_d}) = \frac{e^{-j2\pi f_d}}{3}(1 + 2\cos(2\pi f_d))$$

- Let $x[n]$ be a complex exponential with $f_d = 0$.
  - Then, all samples of $x[n]$ equal to one.
- The output signal $y[n]$ also has all samples equal to one.
- For $f_d = 0$, the frequency response $H(e^{j2\pi 0}) = 1$.
- And, the output $y[n]$ is given by

$$y[n] = H(e^{j2\pi 0}) \cdot \exp(j2\pi 0 n),$$

i.e., all samples are equal to one.

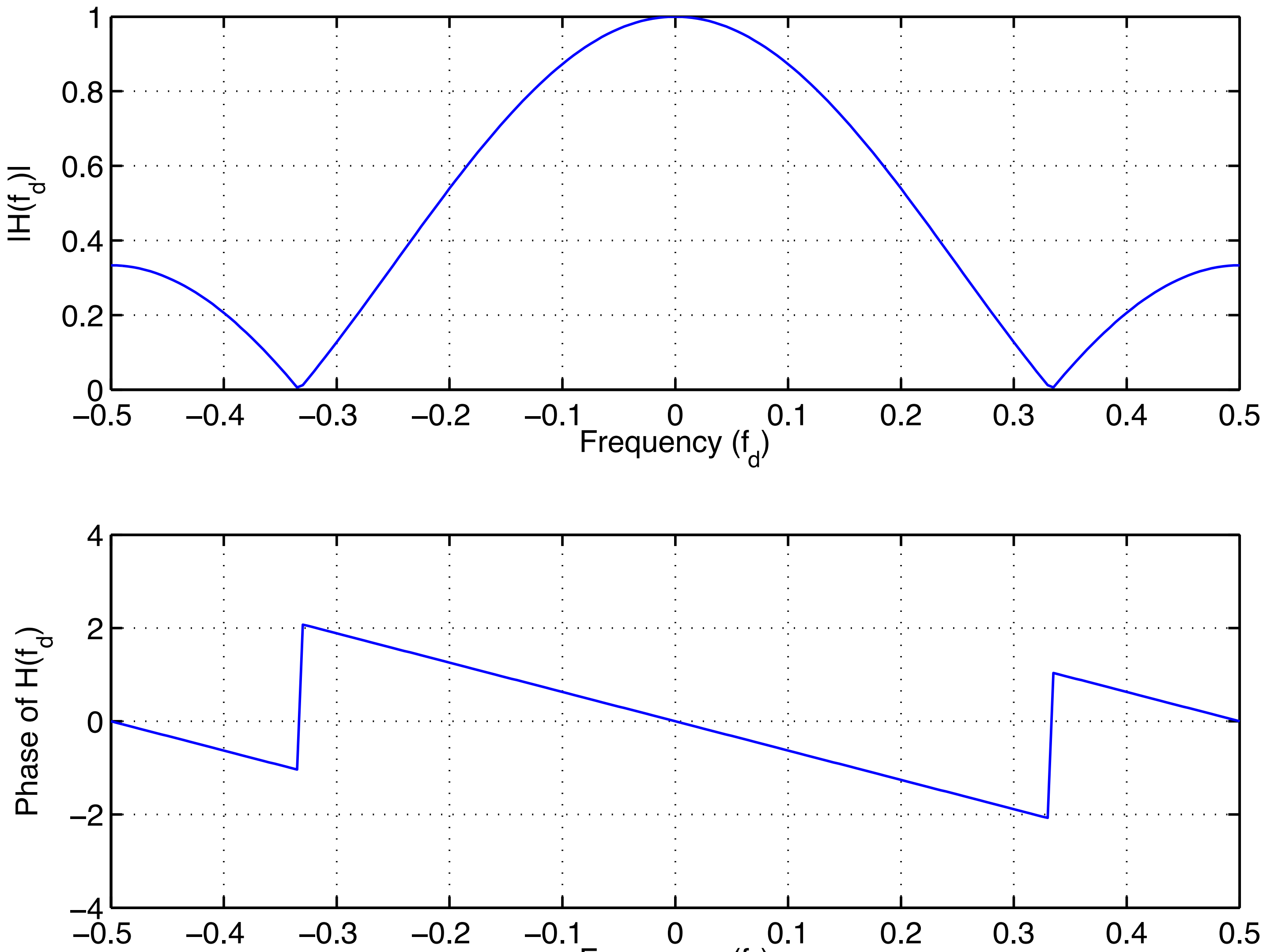
Examples

- Let $x[n]$ be a complex exponential with $f_d = \frac{1}{3}$.
  - Then, the samples of $x[n]$ are the periodic repetition of $\{1, -\frac{1}{2} + \frac{j\sqrt{3}}{2}, -\frac{1}{2} - \frac{j\sqrt{3}}{2}\}$.
- The 3-point average over three consecutive samples equals zero; therefore, $y[n] = 0$.
- For $f_d = \frac{1}{3}$, the frequency response $H(e^{j2\pi f_d}) = 0$.
- Consequently, the output $y[n]$ is given by

$$y[n] = H(\frac{1}{3}) \cdot \exp(j2\pi\frac{1}{3}n) = 0.$$

Thus, all output samples are equal to zero.

Plot of Frequency Response



General Complex Exponential

- Let $x[n]$ be a complex exponential of the from $Ae^{j(2\pi f_d n+\phi)}$.
  - This signal can be written as

$$x[n] = X \cdot e^{j2\pi f_d n},$$

where $X = Ae^{j\phi}$ is the *phasor* of the signal.

- Then, the output $y[n]$ is given by

$$y[n] = H(e^{j2\pi f_d}) \cdot X \cdot \exp(j2\pi f_d n).$$

- **Interpretation:** The output is a complex exponential of the same frequency f_d
- The phasor for the output signal is the product $H(e^{j2\pi f_d}) \cdot X$.

Exercise

Assume that the signal $x[n] = \exp(j2\pi f_d n)$ is input to a 4-point averager.

1. Give a general expression for the output signal and identify the frequenchy response of the system.
2. Compute the output signals for the specific frequencies $f_d = 0$, $f_d = 1/4$, and $f_d = 1/2$.