Overview

- **Today:** a really important, somewhat challenging, class.
- **Key result:** for every linear, time-invariant system (LTI system) the output is obtained from input via convolution.
  - Convolution is a very important operation!
- Prerequisites from previous classes:
  - Impulse signal and impulse response,
  - convolution,
  - linearity, and
  - time-invariance.

Reminders: Convolution and Impulse Response

- **We learned so far:**
  - For FIR filters, input-output relationship

$$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

  - If $x[n] = \delta[n]$, then $y[n] = h[n]$ is called the impulse response of the system.
    - For FIR filters:

$$h[n] = \begin{cases} b_n & \text{for } 0 \leq n \leq M \\ 0 & \text{else.} \end{cases}$$

  - **Convolution:** input-output relationship

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$

Reminders: Linearity and Time-Invariance

- **Linearity:**
  - For arbitrary input signals $x_1[n]$ and $x_2[n]$, let the ouputs be denoted $y_1[n]$ and $y_2[n]$.
  - Further, for the input signal $x[n] = x_1[n] + x_2[n]$, let the output signal be $y[n]$.
  - The system is linear if $y[n] = y_1[n] + y_2[n]$.
- **Time-Invariance:**
  - For an arbitrary input signal $x[n]$, let the output be $y[n]$.
  - For the delayed input $x_d[n] = x[n - n_0]$, let the output be $y_d[n]$.
  - The system is time-invariant if $y_d[n] = y[n - n_0]$.
- **Today:** For any linear, time-invariant system: input-output relationship is $y[n] = x[n] * h[n]$.

Preliminaries

- We need a few more facts and relationships for the impulse signal $\delta[n]$.
- To start, recall:
  - If input to a system is the impulse signal $\delta[n]$,
  - then, the output is called the impulse response,
  - and is denoted by $h[n]$.
- We will derive a method for expressing arbitrary signals $x[n]$ in terms of impulses.

Sifting with Impulses

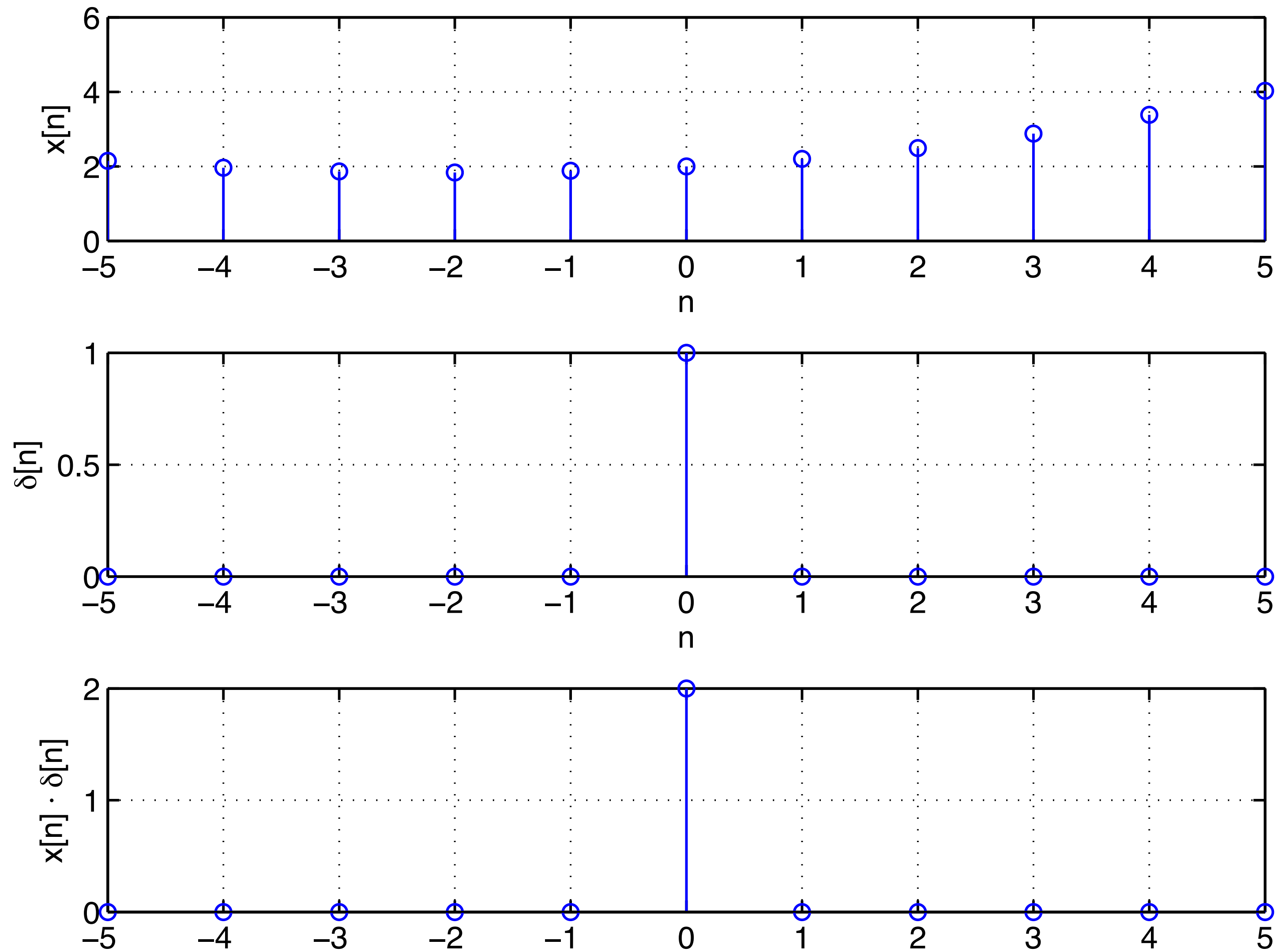
- **Question:** What happens if we multiply a signal $x[n]$ with an impulse signal $\delta[n]$?
- Because

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{else,} \end{cases}$$

- it follows that

$$x[n] \cdot \delta[n] = x[0] \cdot \delta[n] = \begin{cases} x[0] & \text{for } n = 0 \\ 0 & \text{else} \end{cases}$$

Illustration



Sifting with Impulses

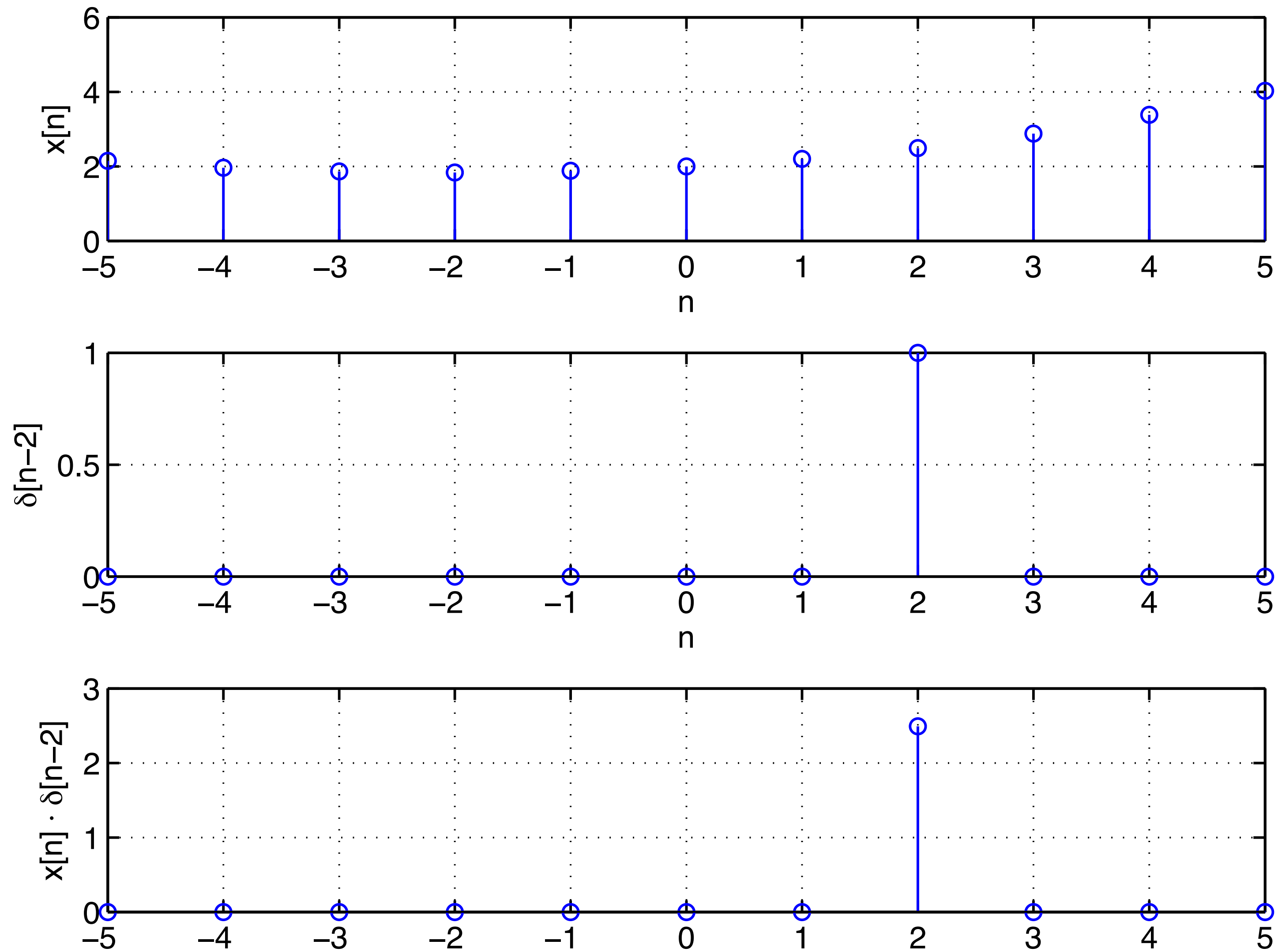
- **Related Question:** What happens if we multiply a signal $x[n]$ with a delayed impulse signal $\delta[n-k]$?
- Recall that $\delta[n-k]$ is an impulse located at the k-th sampling instance:

$$\delta[n-k] = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{else} \end{cases}$$

- It follows that

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k] = \begin{cases} x[k] & \text{for } n = k \\ 0 & \text{else} \end{cases}$$

Illustration



Decomposing a Signal with Impulses

- **Question:** What happens if we combine (add) signals of the form $x[n] \cdot \delta[n-k]$?
- Specifically, what is

$$\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]?$$

  - Notice that the above sum represents the convolution of $x[n]$ and $\delta[n]$, $\delta[n] * x[n]$.

Decomposing a Signal with Impulses

n	...	-1	0	1	2	...
$x[n]$	...	x[-1]	x[0]	x[1]	x[2]	...
$\delta[n]$	...	0	1	0	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x[-1] \cdot \delta[n+1]$	...	x[-1]	0	0	0	...
$x[0] \cdot \delta[n]$	...	0	x[0]	0	0	...
$x[1] \cdot \delta[n-1]$	...	0	0	x[1]	0	...
$x[2] \cdot \delta[n-2]$	...	0	0	0	x[2]	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$	...	x[-1]	x[0]	x[1]	x[2]	...

Decomposing a Signal with Impulses

- From these considerations we conclude that

$$\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] = x[n].$$

  - Notice that this implies

$$x[n] * \delta[n] = x[n].$$

- We now have a way to write a signal $x[n]$ as a sum of scaled and delayed impulses.
- Next, we exploit this relationship to derive our main result.

Applying Linearity and Time-Invariance

- We know already that input $\delta[n]$ produces output $h[n]$ (impulse repsonse). We write:

$$\delta[n] \mapsto h[n].$$

- For a time-invariant system:

$$\delta[n-k] \mapsto h[n-k].$$

- And for a linear system:

$$x[k] \cdot \delta[n-k] \mapsto x[k] \cdot h[n-k].$$

Derivation of the Convolution Sum

- Linearity: linear combination of input signals produces output equal to linear combination of individual outputs.

$$\text{Input} \quad \mapsto \quad \text{Output}$$

$$\begin{array}{rcl} \vdots & \vdots & \vdots \\ x[-1]\cdot\delta[n+1] & \mapsto & x[-1]\cdot h[n+1] \\ x[0]\cdot\delta[n] & \mapsto & x[0]\cdot h[n] \\ x[1]\cdot\delta[n-1] & \mapsto & x[1]\cdot h[n-1] \\ x[2]\cdot\delta[n-1] & \mapsto & x[2]\cdot h[n-2] \\ \vdots & \vdots & \vdots \end{array}$$

$$\sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k] = x[n] \quad \mapsto \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

Summary and Conclusions

- We just derived the convolution sum formula:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$

- We only assumed that the system is linear and time-invariant.
- Therefore, we can conclude that *for any* linear, time-invariant system, the output is the convolution of input and impulse response.
  - Needless to say: convolution and impulse response are enormously important concepts.

Identity System

- From our discussion, we can draw another conclusion.
- **Question:** How can we characterize a LTI system for which the output $y[n]$ is the same as the input $x[n]$.
  - Such a system is called the identity system.
- Specifically, we want the impulse response $h[n]$ of such a system.
- As always, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input.
- Since the ouput is the same as the input for an identity system, we find the impulse response of the identity system

$$h[n] = \delta[n].$$

Ideal Delay Systems

- **Closely Related Question:** How can one characterize a LTI system for which the output $y[n]$ is a delayed version of the input $x[n]$:

$$y[n] = x[n - n_0]$$

where n_0 is the delay introduced by the system

  - Such a system is called an ideal delay system.
- Again, we want the impulse response $h[n]$ of such a system.
- As before, one finds the impulse response $h[n]$ as the output of the LTI system when the impulse $\delta[n]$ is the input.
- Since the ouput is merely a delayed version of the input, we find

$$h[n] = \delta[n - n_0].$$