The Sampling Theorem

- We have analyzed the relationship between the frequency f of a sinusoid and the sampling rate f_s.
  - We saw that the ratio f/f_s must be less than $1/2$, i.e., $f_s > 2 \cdot f$. Otherwise aliasing or folding occurs.
- This insight provides the first half of the famous sampling theorem

> A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2 \cdot f_{max}$.

- This very import result is attributed to Claude Shannon and Harry Nyquist.

Reconstructing a Signal from Samples

- The sampling theorem suggests that the original continuous-time signal $x(t)$ can be recreated from its samples $x[n]$.
  - Assuming that samples were taken at a high enough rate.
  - This process is referred to as reconstruction or D-to-C conversion (discrete-time to continuous-time conversion).
- In principle, the continous-time signal is reconstructed by placing a suitable pulse at each sample location and adding all pulses.
  - The amplitude of each pulse is given by the sample value.

Suitable Pulses

- Suitable pulses include
  - Rectangular pulse (zero-order hold):

$$p(t) = \begin{cases} 1 & \text{for } -T_s/2 \leq t < T_s/2 \\ 0 & \text{else.} \end{cases}$$

  - Triangular pulse (linear interpolation)

$$p(t) = \begin{cases} 1 + t/T_s & \text{for } -T_s \leq t \leq 0 \\ 1 - t/T_s & \text{for } 0 \leq t \leq T_s \\ 0 & \text{else.} \end{cases}$$

Reconstruction

- The reconstructed signal $\hat{x}(t)$ is computed from the samples and the pulse $p(t)$:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot p(t - nT_s).$$

- The reconstruction formula says:
  - place a pulse at each sampling instant ($p(t - nT_s)$),
  - scale each pulse to amplitude $x[n]$,
  - add all pulses to obtain the reconstructed signal.

Ideal Reconstruction

- Reconstruction with the above pulses will be pretty good.
  - Particularly, when the sampling rate is much greater than twice the signal frequency (significant oversampling).
- However, reconstruction is not perfect as suggested by the sampling theorem.
- To obtain perfect reconstruction the following pulse must be used:

$$p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}.$$

- This pulse is called the sinc pulse.
- Note, that it is of infinite duration and, therefore, is not practical.
  - In practice a truncated version may be used for excellent reconstruction.

The sinc pulse