Signal Operations in the Frequency Domain

- Signal processing implies that we apply *operations* to signals; Examples include:
  - Adding two signals
  - Delaying a signal
  - Multiplying a signal with a complex exponential signal
- **Question:** What does each of these operation do the spectrum of the signal?
  - We will answer that question for some common signal processing operations.

Scaling a Signal

- Let $x(t)$ be a signal with spectrum $X(f) = \{(X_n, f_n)\}_n$.
- **Question:** If c is a scalar constant, what is the spectrum of the signal $y(t) = c \cdot x(t)$?
- Since

$$x(t) = \sum_n X_n \cdot e^{j2\pi f_n t}$$

$$y(t) = c \cdot x(t) = \sum_n c \cdot X_n \cdot e^{j2\pi f_n t}.$$

- Therefore,

$$Y(f) = \{(c \cdot X_n, f_n)\}_n.$$

  - We use the short-hand $Y(f) = c \cdot X(f)$ to denote $\{(c \cdot X_n, f_n)\}_n$.

Adding Two Signals

- Let $x(t)$ and $y(t)$ be signals with spectra $X(f)$ and $Y(f)$.
- **Question:** What is the spectrum of the signal $z(t) = x(t) + y(t)$?
- Since

$$z(t) = x(t) + y(t) = \sum_n X_n \cdot e^{j2\pi f_n t} + \sum_n Y_n \cdot e^{j2\pi f_n t}$$

$$Z(f) = \{(X_n + Y_n, f_n)\}_n.$$

  - We use the short-hand $Z(f) = X(f) + Y(f)$ to denote $\{(X_n + Y_n, f_n)\}$.
- **Example:** What is the spectrum $Z(f)$ when signals with spectra $X(f) = \{(3,0), (1,1), (1,-1), (2,2), (2,-2)\}$ and $Y(f) = \{(j,1), (-j,-1), (1,3), (1,-3)\}$ are added?

Delaying a Signal

- Let $x(t)$ be a signal and $X(f) = \{(X_n, f_n)\}_n$ denotes its spectrum.
- **Question:** What is the spectrum of the signal $y(t) = x(t - \tau)$?
- Since

$$y(t) = x(t-\tau) = \sum_n X_n \cdot e^{j2\pi f_n (t-\tau)} = \sum_n X_n e^{-j2\pi f_n \tau} \cdot e^{j2\pi f_n t}$$

it follows that

$$Y(f) = \{(X_n e^{-j2\pi f_n \tau}, f_n)\}_n.$$

  - Notice that delaying a signal induces *phase shifts* in the spectrum
  - The phase shifts are proportional to the delay τ and the frequencies f_n.

Delaying a Signal – Example

- **Example:** What is the spectrum $Y(f)$ when the signal with spectrum $X(f) = \{(3,0),(1,1),(1,-1),(2,2),(2,-2)\}$ is shifted by $\tau = \frac{1}{4}$?
- **Answer:**

$$Y(f) = \{(3,0),(-j,1),(j,-1),(-2,2),(-2,-2)\}$$

Multiplying by a Complex Exponential

- Let $x(t)$ be a signal and $X(f) = \{(c \cdot X_n, f_n)\}_n$ denotes its spectrum.
- **Question:** What is the spectrum of the signal $y(t) = x(t) \cdot e^{j2\pi f_c t}$?
- Since

$$y(t) = x(t) \cdot e^{j2\pi f_c t} = \sum_n X_n \cdot e^{j2\pi f_n t} \cdot e^{j2\pi f_c t} = \sum_n X_n \cdot e^{j2\pi (f_n + f_c) t}$$

it follows that

$$Y(f) = \{X_n, f_n + f_c\}$$

  - Notice that the entire spectrum is shifted by f_c, i.e., $Y(f) = X(f + f_c)$.
  - Notice the "symmetry" with the time delay operation — this is called duality.

Exercise: Spectrum of AM Signal

- We discussed that amplitude modulation *processess* a message signal to produce the transmitted signal $s(t)$:

$$s(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

- Assume that the spectrum of $m(t)$ is $M(f)$.
- **Question:** Use the Spectrum Operations we discussed to express the spectrum $S(f)$ in terms of $M(f)$.
- **Answer:**

$$S(f) = \frac{1}{2}M(f + f_c) + \frac{1}{2}M(f - f_c) + \{(\frac{A}{2}, f_c) + \{(\frac{A}{2}, -f_c)\}$$