Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- *Ingredients* are given by the spectrum

$$X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\}$$

  Each pair indicates one complex exponential component by listing its frequency and complex amplitude.
- *Instructions* for combining the ingredients and producing the (time-domain) signal:

$$x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t).$$

- Always simplify the expression you obtain!

Example

- Problem: Find the signal $x(t)$ corresponding to

$$X(f) = \{(3,0), \quad (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}$$

- Solution:

$$x(t) = 3 \quad + \tfrac{5}{2}e^{-j\pi/2}e^{j2\pi 10t} + \tfrac{5}{2}e^{j\pi/2}e^{-j2\pi 10t} \\ + \tfrac{7}{2}e^{j\pi/4}e^{j2\pi 25t} + \tfrac{7}{2}e^{-j\pi/4}e^{-j2\pi 25t}$$

- Which simplifies to:

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

Exercise

- Find the signal with the spectrum:

$$X(f) = \{(5,0), \quad (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \\ (\tfrac{5}{2}e^{j\pi/4}, 15), (\tfrac{5}{2}e^{-j\pi/4}, -15)\}$$

Analysis: From Time to Frequency-Domain

- The objective of spectrum or Fourier analysis is to find the spectrum of a time-domain signal.
- We will restrict ourselves to signals $x(t)$ that are sums of sinusoids

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- We have already shown that such signals have spectrum:

$$X(f) = \{(X_0, 0), (\frac{1}{2}X_1, f_1), (\frac{1}{2}X_1^*, -f_1), \ldots, (\frac{1}{2}X_N, f_N), (\frac{1}{2}X_N^*, -f_N)$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

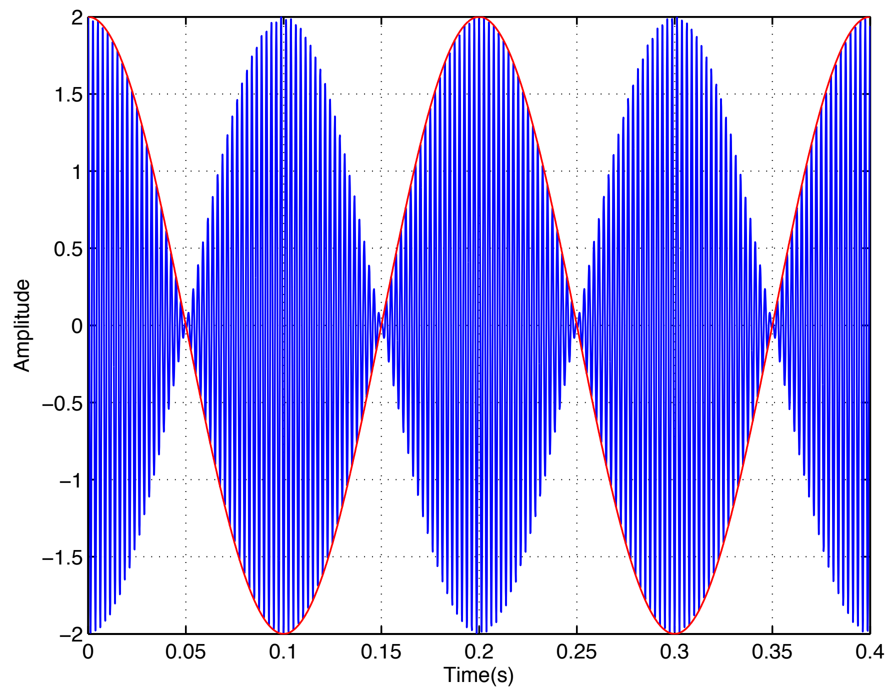
- We will investigate some interesting signals that can be written as a sum of sinusoids.

Beat Notes

- Consider the signal

$$x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t).$$

- This signal does not have the form of a sum of sinusoids; hence, we can not determine it's spectrum immediately.



MATLAB Code for Beat Notes

```
% Parameters
fs  = 8192;
dur = 2;

f1 = 5;
f2 = 400;
A  = 2;

NP  = round(2*fs/f1); % number of samples to plot

% time axis and signal
tt=0:1/fs:dur;
xx = A*cos(2*pi*f1*tt).*cos(2*pi*f2*tt);

plot(tt(1:NP),xx(1:NP),tt(1:NP),A*cos(2*pi*f1*tt(1:NP)),'r')
xlabel('Time(s)')
ylabel('Amplitude')
grid
```

Beat Notes as a Sum of Sinusoids

- Using the inverse Euler relationships, we can write

$$\begin{aligned} x(t) &= 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t) \\ &= 2 \cdot \tfrac{1}{2} \cdot (e^{j2\pi 5t} + e^{-j2\pi 5t}) \cdot \tfrac{1}{2} \cdot (e^{j2\pi 400t} + e^{-j2\pi 400t}). \end{aligned}$$

- Multiplying out yields:

$$x(t) = \frac{1}{2}(e^{j2\pi 405t} + e^{-j2\pi 405t}) + \frac{1}{2}(e^{j2\pi 395t} + e^{-j2\pi 395t}).$$

- Applying Euler's relationship, lets us write:

$$x(t) = \cos(2\pi 405t) + \cos(2\pi 395t).$$

Spectrum of Beat Notes

- We were able to rewrite the beat notes as a sum of sinusoids

$$x(t) = \cos(2\pi 405 t) + \cos(2\pi 395 t).$$

- Note that the frequencies in the sum, 395 Hz and 405 Hz, are the sum and difference of the frequencies in the original product, 5 Hz and 400 Hz.
- It is now straightforward to determine the spectrum of the beat notes signal:

$$X(f) = \{(\frac{1}{2}, 405), (\frac{1}{2}, -405), (\frac{1}{2}, 395), (\frac{1}{2}, -395)\}$$

Spectrum of Beat Notes