Lecture: The Phasor Addition Rule

Problem Statement

- It is often required to add two or more sinusoidal signals.
- When all sinusoids have the same frequency then the problem simplifies.
  - This problem comes up very often, e.g., in AC circuit analysis (ECE 280) and later in the class (chapter 5).
- Starting point: sum of sinusoids

$$x(t) = A_1 \cos(2\pi f t + \phi_1) + \ldots + A_N \cos(2\pi f t + \phi_N)$$

  - Note that all frequencies f are the same (no subscript).
  - Amplitudes A_i phases ϕ_i are different in general.
  - Short-hand notation using summation symbol ($\sum$):

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i)$$

The Phasor Addition Rule

- The phasor addition rule implies that there exist an amplitude A and a phase ϕ such that

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) = A \cos(2\pi f t + \phi)$$

- **Interpretation:** The sum of sinusoids of the same frequency but different amplitudes and phases is
  - a single sinusoid of the same frequency.
  - The phasor addition rule specifies how the amplitude A and the phase ϕ depends on the original amplitudes A_i and ϕ_i.
- **Example:** We showed earlier (by means of an unpleasant computation involving trig identities) that:

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2) = 5 \cos(2\pi f t + 53^o)$$

Prerequisites

- We will need two simple prerequisites before we can derive the phasor addition rule.
  1. Any sinusoid can be written in terms of complex exponentials as follows

     $$A\cos(2\pi ft+\phi) = \mathrm{Re}\{Ae^{j(2\pi ft+\phi)}\} = \mathrm{Re}\{Ae^{j\phi}e^{j2\pi ft}\}.$$

     Recall that $Ae^{j\phi}$ is called a phasor (complex amplitude).
  2. For any complex numbers $X_1, X_2, \ldots, X_N$, the real part of the sum equals the sum of the real parts.

     $$\mathrm{Re}\left\{\sum_{i=1}^{N} X_i\right\} = \sum_{i=1}^{N}\mathrm{Re}\{X_i\}.$$

     - This should be obvious from the way addition is defined for complex numbers.

       $$(x_1+jy_1)+(x_2+jy_2) = (x_1+x_2)+j(y_1+y_2).$$

Deriving the Phasor Addition Rule

- **Objective**: We seek to establish that

$$\sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) = A \cos(2\pi ft + \phi)$$

and determine how A and ϕ are computed from the A_i and ϕ_i.

Deriving the Phasor Addition Rule

- **Step 1:** Using the first pre-requisite, we replace the sinusoids with complex exponentials

$$\begin{aligned}\sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i) &= \sum_{i=1}^{N} \text{Re}\{A_i e^{j(2\pi ft + \phi_i)}\} \\ &= \sum_{i=1}^{N} \text{Re}\{A_i e^{j\phi_i} e^{j2\pi ft}\}.\end{aligned}$$

Deriving the Phasor Addition Rule

- **Step 2:** The second prerequisite states that the sum of the real parts equals the the real part of the sum

$$\sum_{i=1}^{N} \mathrm{Re}\{A_i e^{j\phi_i} e^{j2\pi ft}\} = \mathrm{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\}.$$

Deriving the Phasor Addition Rule

- **Step 3:** The exponential $e^{j2\pi ft}$ appears in all the terms of the sum and can be factored out

$$\text{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\} = \text{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\}$$

  - The term $\sum_{i=1}^{N} A_i e^{j\phi_i}$ is just the sum of complex numbers in polar form.
  - The sum of complex numbers is just a complex number X which can be expressed in polar form as $X = Ae^{j\phi}$.
  - Hence, amplitude A and phase ϕ must satisfy

$$Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$$

Deriving the Phasor Addition Rule

- Note
  - computing $\sum_{i=1}^{N} A_i e^{j\phi_i}$ requires converting $A_i e^{j\phi_i}$ to rectangular form,
  - the result will be in rectangular form and must be converted to polar form $Ae^{j\phi}$.

Deriving the Phasor Addition Rule

- **Step 4:** Using $Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$ in our expression for the sum of sinusoids yields:

$$
\begin{aligned}
\mathrm{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\} &= \mathrm{Re}\left\{Ae^{j\phi} e^{j2\pi ft}\right\} \\
&= \mathrm{Re}\left\{Ae^{j(2\pi ft+\phi)}\right\} \\
&= A\cos(2\pi ft + \phi).
\end{aligned}
$$

- Note: the above result shows that the sum of sinusoids of the same frequency is a sinusoid of the same frequency.

Applying the Phasor Addition Rule

- Applicable only when sinusoids of same frequency need to be added!
- **Problem:** Simplify

$$x(t) = A_1 \cos(2\pi ft + \phi_1) + \ldots A_N \cos(2\pi ft + \phi_N)$$

- **Solution:** proceeds in 4 steps
  1. Extract phasors: $X_i = A_i e^{j\phi_i}$ for $i = 1, \ldots, N$.
  2. Convert phasors to rectangular form: $X_i = A_i \cos\phi_i + jA_i \sin\phi_i$ for $i = 1, \ldots, N$.
  3. Compute the sum: $X = \sum_{i=1}^{N} X_i$ by adding real parts and imaginary parts, respectively.
  4. Convert result X to polar form: $X = Ae^{j\phi}$.
- **Conclusion:** With amplitude A and phase ϕ determined in the final step

$$x(t) = A\cos(2\pi ft + \phi).$$

Example

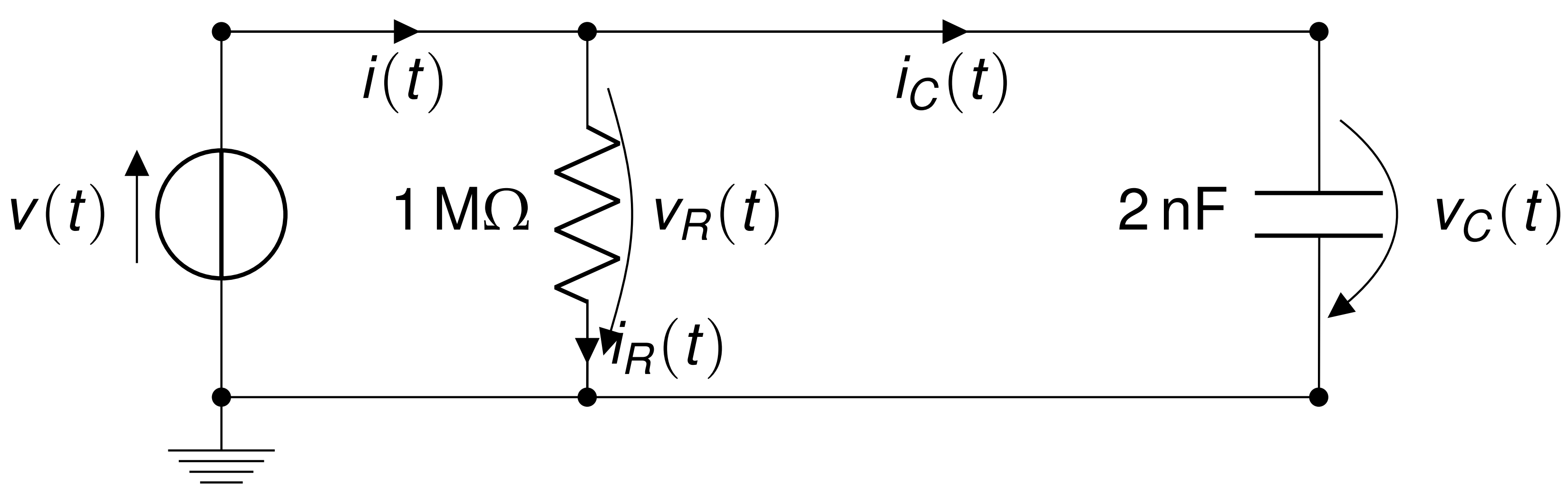
- **Problem:** Simplify

$$x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi/2)$$

- **Solution:**
  1. Extract Phasors: $X_1 = 3e^{j0} = 3$ and $X_2 = 4e^{j\pi/2}$.
  2. Convert to rectangular form: $X_1 = 3$ $X_2 = 4j$.
  3. Sum: $X = X_1 + X_2 = 3 + 4j$.
  4. Convert to polar form: $A = \sqrt{3^2 + 4^2} = 5$ and $\phi = \arctan(\frac{4}{3}) \approx 53^o$ $(\frac{53}{180}\pi)$.
- **Result:**

$$x(t) = 5\cos(2\pi ft + 53^o).$$

The Circuits Example



- For $v(t) = 1\,\text{V} \cdot \cos(2\pi 1\,\text{kHz} \cdot t)$, find the current $i(t)$.

Problem Formulation with Phasors

- Source:

$$v(t) = 1\,\mathrm{V} \cdot \cos(2\pi 1\,\mathrm{kHz} \cdot t) = \mathrm{Re}\{1\,\mathrm{V} \cdot \exp(j2\pi 1\,\mathrm{kHz} \cdot t)\}$$

  $\Rightarrow$ phasor: $V = 1\,\mathrm{V} e^{j0}$
- Kirchhoff's voltage law: $v(t) = v_R(t) = v_C(t)$;
  $\Rightarrow$ phasors: $V = V_R = V_C$.
- Resistor: $i_R(t) = \frac{v_R(t)}{R}$;
  $\Rightarrow$ phasor: $I_R = \frac{V_R}{R}$
- Capacitor: $i_C(t) = C\frac{dv_C(t)}{dt}$;
  $\Rightarrow$ phasor: $I_C = C \cdot V \cdot j2\pi \cdot 1\,\mathrm{kHz}$
  - Because $\frac{d\,\exp(j2\pi 1\,\mathrm{kHz} \cdot t)}{dt} = j2\pi 1\,\mathrm{kHz} \cdot \exp(j2\pi 1\,\mathrm{kHz} \cdot t)$
- Kirchhoff's current law: $i(t) = i_R(t) + i_C(t)$;
  $\Rightarrow$ phasors: $I = I_R + I_C$.

Problem Formulation with Phasors

- Therefore,

$$I = \frac{V}{R} + C \cdot V \cdot j2\pi \cdot 1\,\text{kHz}$$
$$= \frac{1\,\text{V}}{1\,\text{M}\Omega} + j2\pi \cdot 1\,\text{kHz} \cdot 2\,\text{nF} \cdot 1\,\text{V}$$
$$= 1\,\mu\text{A} + j4\pi\,\mu\text{A}$$

- Convert to polar form:

$$1\,\mu\text{A} + j4\pi\,\mu\text{A} = 12.6\,\mu\text{A} \cdot e^{j0.47\pi}$$

  Using:
  - $\sqrt{1^2 + (4\pi)^2} \approx 12.6$
  - $\tan^{-1}((4\pi)) \approx 0.47\pi$
- Thus, $i(t) \approx 12.6\,\mu\text{A}\cos(2\pi 1\,\text{kHz} \cdot t + 0.47 \cdot \pi)$.

Exercise

- Simplify

$$x(t) = 10\cos(20\pi t + \frac{\pi}{4}) + \\ 10\cos(20\pi t + \frac{3\pi}{4}) + \\ 20\cos(20\pi t - \frac{3\pi}{4}).$$

- Answer:

$$x(t) = 10\sqrt{2}\cos(20\pi t + \pi).$$