Lecture: Complex Exponentials
Introduction

- The complex exponential signal is defined as

\[ x(t) = A \exp(j(2\pi ft + \phi)). \]

- As with sinusoids, \( A, f, \) and \( \phi \) are (real-valued) amplitude, frequency, and phase.

- By Euler’s relationship, it is closely related to sinusoidal signals

\[ x(t) = A \cos(2\pi ft + \phi) + jA \sin(2\pi ft + \phi). \]

- We will leverage the benefits the complex representation provides over sinusoids:
  - Avoid trigonometry,
  - Replace with simple algebra,
  - Visualization in the complex plane.
Plot of Complex Exponential

\[ x(t) = 1 \cdot \exp(j(2\pi/8t + \pi/4)) \]

Since \( x(t) \) is complex-valued, both real and imaginary parts are functions of time.
We can think of a complex exponential as signals that rotate along a circle in the complex plane.

\[ x(t) = 1 \cdot e^{j(2\pi/8t + \pi/4)} \]
Expressing Sinusoids through Complex Exponentials

- There are two ways to write a sinusoidal signal in terms of complex exponentials.
- **Real part:**
  \[ A \cos(2\pi ft + \phi) = \text{Re}\{A \exp(j(2\pi ft + \phi))\} \].
- **Inverse Euler:**
  \[ A \cos(2\pi ft + \phi) = \frac{A}{2}(\exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi))) \]
- Both expressions are useful and will be important throughout the course.
Phasors

- Phasors are **not** directed-energy weapons first seen in the original Star Trek movie.
  - That would be *phasers*!
- Phasors are the **complex amplitudes** of complex exponential signals:
  \[ x(t) = A \exp(j(2\pi ft + \phi)) = Ae^{j\phi} \exp(j2\pi ft). \]
- The phasor of this complex exponential is \( X = Ae^{j\phi} \).
- Thus, phasors capture both amplitude \( A \) and phase \( \phi \) – in polar coordinates.
  - The real and imaginary parts of the phasor \( X = Ae^{j\phi} \) are referred to as the *in-phase* (I) and *quadrature* (Q) components of \( X \), respectively:
  \[ X = I + jQ = A \cos(\phi) + jA \sin(\phi) \]
Phasor Notation for Complex Exponentials

- The complex exponential signal

\[ x(t) = A \exp(j(2\pi ft + \phi)) = Ae^{j\phi} \exp(j2\pi ft) \]

is characterized completely by the combination of

- phasor \( X = Ae^{j\phi} \)
- frequency \( f \)

- We will frequently use this observation to denote a complex exponential by providing the pair of phasor and frequency:

\[(Ae^{j\phi}, f)\]

- We will refer to this notation as the *spectrum representation* of the complex exponential \( x(t) \)
From Sinusoids to Phasors

- A sinusoid can be written as

\[ A \cos(2\pi ft + \phi) = \frac{A}{2} (\exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi))). \]

- This can be rewritten to provide

\[ A \cos(2\pi ft + \phi) = \frac{Ae^{j\phi}}{2} \exp(j2\pi ft) + \frac{Ae^{-j\phi}}{2} \exp(-j2\pi ft). \]

- Thus, a sinusoid is composed of two complex exponentials

  - One with frequency \( f \) and phasor \( \frac{Ae^{j\phi}}{2} \),
    - rotates counter-clockwise in the complex plane;
  - one with frequency \( -f \) and phasor \( \frac{Ae^{-j\phi}}{2} \).
    - rotates clockwise in the complex plane;

- Note that the two phasors are conjugate complexes of each other.
Exercise

- Write

\[ x(t) = 3 \cos(2\pi 10t - \pi/3) \]

as a sum of two complex exponentials.

- For each of the two complex exponentials, find the frequency and the phasor.

- Repeat for

\[ y(t) = 2 \sin(2\pi 10t + \pi/4) \]

- What are the in-phase and quadrature signals of

\[ z(t) = 5e^{j\pi/3} \exp(j2\pi 10t) \]
Answers to Exercise

\[ x(t) = 3 \cos(2\pi 10t - \pi/3) \]
\[ = \frac{3}{2} e^{-j\pi/3} e^{j2\pi 10t} + \frac{3}{2} e^{j\pi/3} e^{-j2\pi 10t} \]

as a sum of two complex exponentials.

Phasor-frequency pairs: \((\frac{3}{2} e^{-j\pi/3}, 10)\) and \((\frac{3}{2} e^{j\pi/3}, -10)\)

\[ y(t) = 2 \sin(2\pi 10t + \pi/4) = 2 \cos(2\pi 10t - \pi/4) \]
\[ = 1 e^{-j\pi/4} e^{j2\pi 10t} + 1 e^{j\pi/4} e^{-j2\pi 10t} \]

\[ z(t) = 5 e^{j\pi/3} \exp(j2\pi 10t) = \left( \frac{5}{2} + j \frac{5\sqrt{2}}{2} \right) \exp(j2\pi 10t) \]

Thus, \( I = 5 \) and \( Q = 5\sqrt{2} \)