Lecture: Complex Exponentials

Introduction

- The complex exponential signal is defined as

$$x(t) = A\exp(j(2\pi ft + \phi)).$$

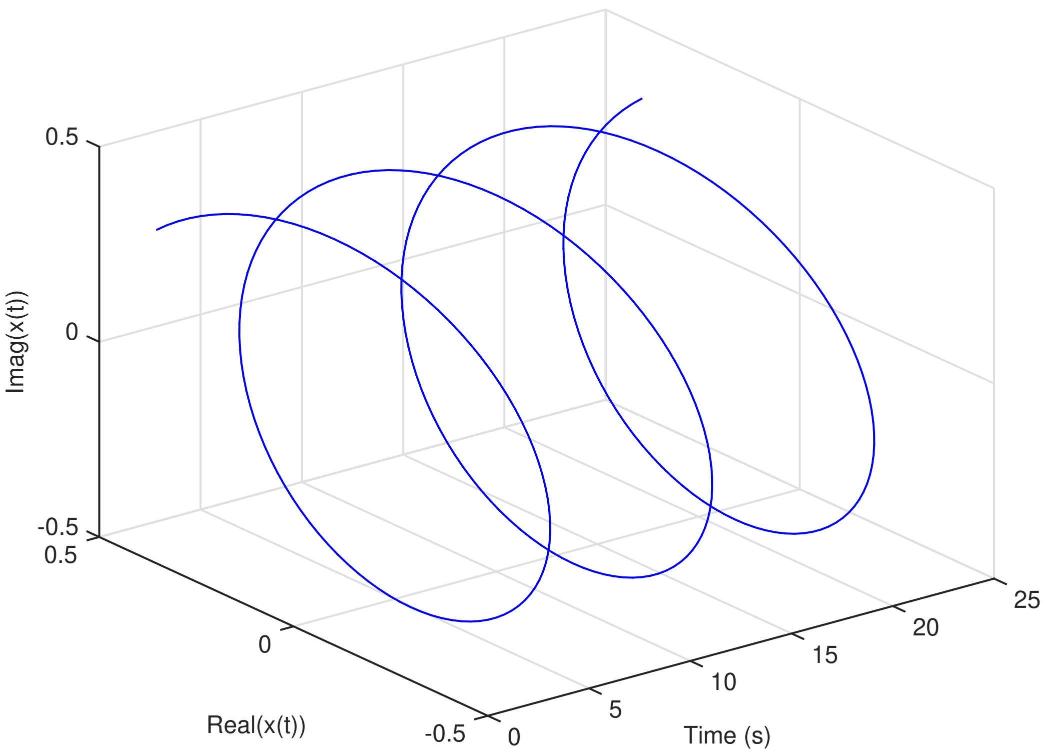
  - As with sinusoids, A, f, and ϕ are (real-valued) amplitude, frequency, and phase.
- By Euler's relationship, it is closely related to sinusoidal signals

$$x(t) = A\cos(2\pi ft + \phi) + jA\sin(2\pi ft + \phi).$$

- We will leverage the benefits the complex representation provides over sinusoids:
  - Avoid trigonometry,
  - Replace with simple algebra,
  - Visualization in the complex plane.

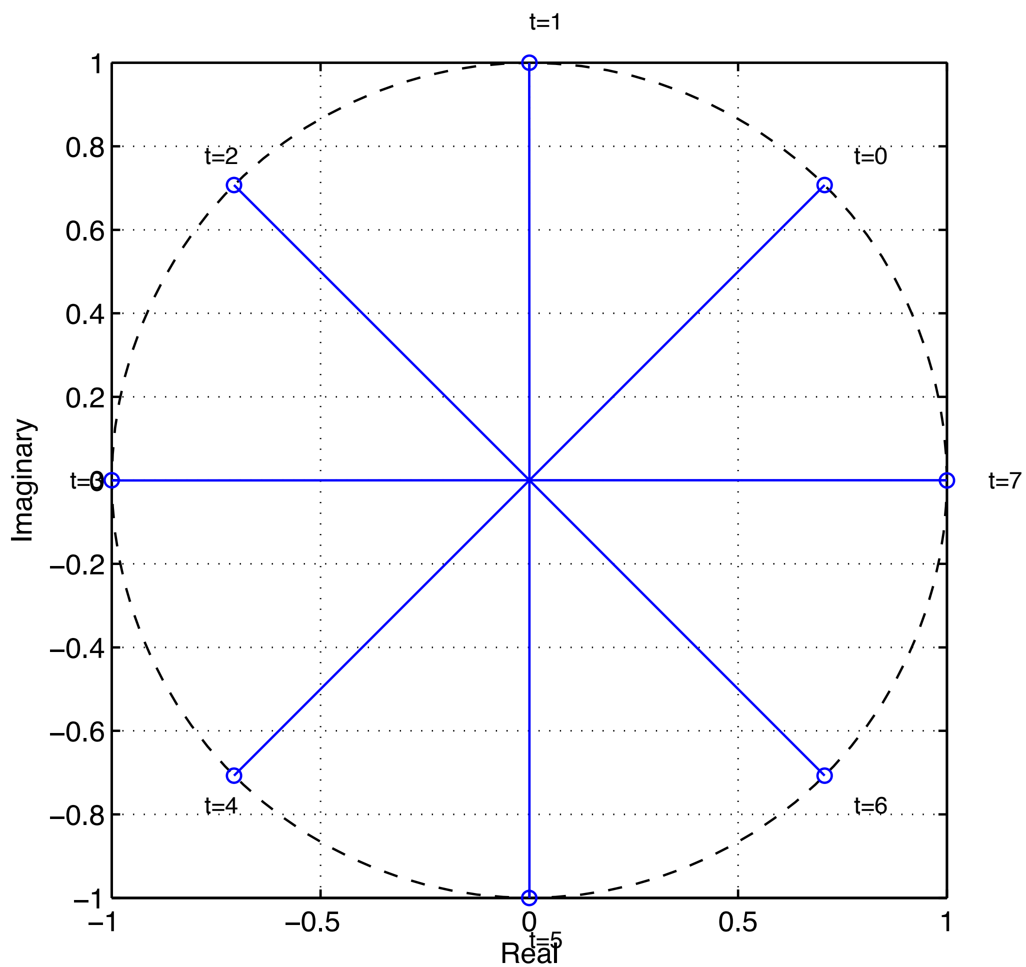
Plot of Complex Exponential

$$x(t) = 1 \cdot \exp(j(2\pi/8t + \pi/4))$$



Since $x(t)$ is complex-valued, both real and imaginary parts are functions of time.

Complex Plane



$$x(t) = 1 \cdot e^{j(2\pi/8t+\pi/4)}$$

We can think of a complex expontial as signals that rotate along a circle in the complex plane.

Expressing Sinusoids through Complex Exponentials

- There are two ways to write a sinusoidal signal in terms of complex exponentials.
- **Real part:**

$$A\cos(2\pi ft + \phi) = \text{Re}\{A\exp(j(2\pi ft + \phi))\}.$$

- **Inverse Euler:**

$$A\cos(2\pi ft + \phi) = \frac{A}{2}(\exp(j(2\pi ft + \phi)) + \exp(-j(2\pi ft + \phi)))$$

- Both expressions are useful and will be important throughout the course.

Phasors

- Phasors are **not** directed-energy weapons first seen in the original Star Trek movie.
  - That would be *phas**e**rs*!
- Phasors are the complex amplitudes of complex exponential signals:

$$x(t) = A\exp(j(2\pi ft + \phi)) = Ae^{j\phi}\exp(j2\pi ft).$$

- The phasor of this complex exponential is $X = Ae^{j\phi}$.
- Thus, phasors capture both amplitude A and phase ϕ – in polar coordinates.
  - The real and imaginary parts of the phasor $X = Ae^{j\phi}$ are referred to as the *in-phase* (I) and *quadrature* (Q) components of X, respectively:

$$X = I + jQ = A\cos(\phi) + jA\sin(\phi)$$

Phasor Notation for Complex Exponentials

- The complex exponential signal

$$x(t) = A\exp(j(2\pi ft + \phi)) = Ae^{j\phi}\exp(j2\pi ft)$$

  is characterized completely by the combination of
  - phasor $X = Ae^{j\phi}$
  - frequency f
- We will frequently use this observation to denote a complex exponential by providing the pair of phasor and frequency:

$$(Ae^{j\phi}, f)$$

  - We will refer to this notation as the *spectrum representation* of the complex exponential $x(t)$

From Sinusoids to Phasors

- A sinusoid can be written as

$$A\cos(2\pi ft+\phi)=\frac{A}{2}(\exp(j(2\pi ft+\phi))+\exp(-j(2\pi ft+\phi))).$$

- This can be rewritten to provide

$$A\cos(2\pi ft+\phi)=\frac{Ae^{j\phi}}{2}\exp(j2\pi ft)+\frac{Ae^{-j\phi}}{2}\exp(-j2\pi ft).$$

- Thus, a sinusoid is composed of two complex exponentials
  - One with frequency f and phasor $\frac{Ae^{j\phi}}{2}$,
    - rotates counter-clockwise in the complex plane;
  - one with frequency $-f$ and phasor $\frac{Ae^{-j\phi}}{2}$.
    - rotates clockwise in the complex plane;
  - Note that the two phasors are conjugate complexes of each other.

Exercise

- Write
$$x(t) = 3\cos(2\pi 10t - \pi/3)$$
as a sum of two complex exponentials.
- For each of the two complex exponentials, find the frequency and the phasor.
- Repeat for
$$y(t) = 2\sin(2\pi 10t + \pi/4)$$
- What are the in-phase and quadrature signals of
$$z(t) = 5e^{j\pi/3}\exp(j2\pi 10t)$$

Answers to Exercise

- $$x(t) = 3\cos(2\pi 10t - \pi/3) = \frac{3}{2}e^{-j\pi/3}e^{j2\pi 10t} + \frac{3}{2}e^{j\pi/3}e^{-j2\pi 10t}$$
  as a sum of two complex exponentials.
- Phasor-frequency pairs: $(\frac{3}{2}e^{-j\pi/3}, 10)$ and $(\frac{3}{2}e^{j\pi/3}, -10)$
- $$y(t) = 2\sin(2\pi 10t + \pi/4) = 2\cos(2\pi 10t - \pi/4) = 1e^{-j\pi/4}e^{j2\pi 10t} + 1e^{j\pi/4}e^{-j2\pi 10t}$$
- $$z(t) = 5e^{j\pi/3}\exp(j2\pi 10t) = (\frac{5}{2} + j\frac{5\sqrt{2}}{2})\exp(j2\pi 10t)$$
  Thus, $I = \frac{5}{2}$ and $Q = \frac{5\sqrt{2}}{2}$