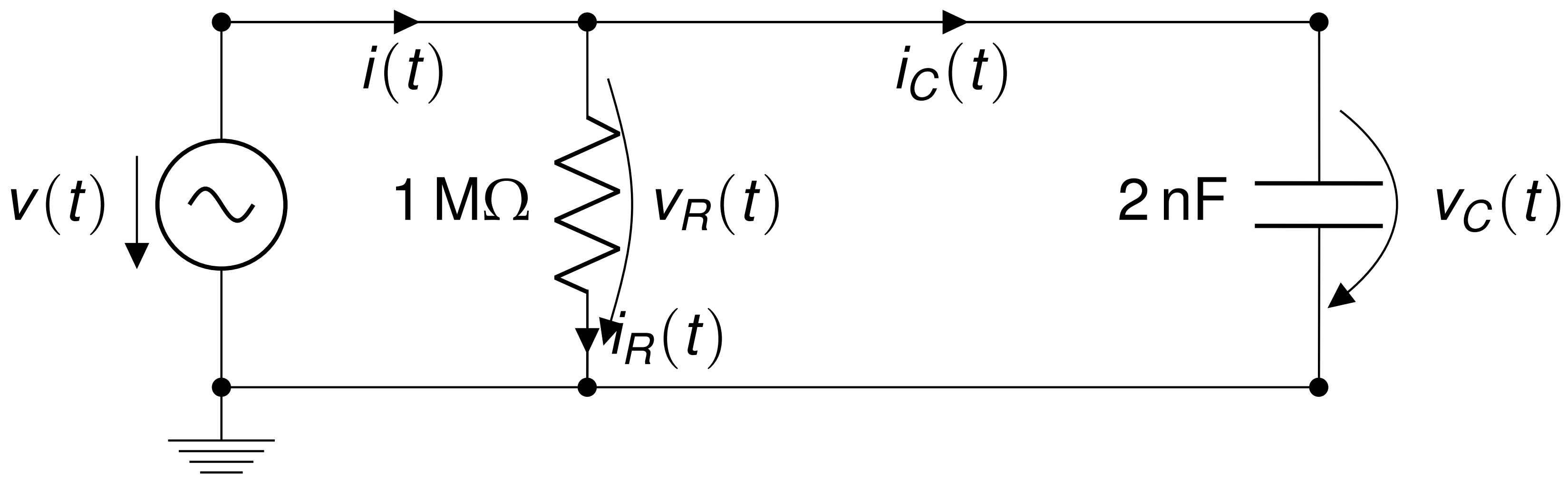
Lecture: Adding Sinusoids of the Same Frequency

Adding Sinusoids

- Adding sinusoids of the same frequency is a problem that arises regularly in
  - circuit analysis
  - linear, time-invariant systems, e.g., filters
  - and many other domains
- We will see that adding sinusoids is much easier with complex exponentials
  - Today, we will do it the hard way — with trigonometry

A Circuits Example



- For $v(t) = 1\,\mathrm{V} \cdot \cos(2\pi 1\,\mathrm{kHz} \cdot t)$, find the current $i(t)$.

Setting up the Problem

- Resistor: $i_R(t) = \frac{v_R(t)}{R}$
- Capacitor: $i_C(t) = C\frac{dv_C(t)}{dt}$
- Kirchhoff's current law: $i(t) = i_R(t) + i_C(t)$
- Kirchhoff's voltage law: $v(t) = v_R(t) = v_C(t)$
- Therefore,

$$
\begin{aligned}
i(t) &= \frac{v(t)}{R} + C \cdot \frac{dv(t)}{dt} \\
&= \frac{1\,\mathrm{V}}{1\,\mathrm{M\Omega}} \cos(2\pi 1\,\mathrm{kHz} \cdot t) - 2\pi \cdot 1\,\mathrm{kHz} \cdot 2\,\mathrm{nF} \cdot \sin(2\pi 1\,\mathrm{kHz} \cdot t) \\
&= 1\,\mu\mathrm{A} \cos(2\pi 1\,\mathrm{kHz} \cdot t) - 4\pi\,\mu\mathrm{A} \sin(2\pi 1\,\mathrm{kHz} \cdot \mathrm{t})
\end{aligned}
$$

Simplifying $i(t)$

- Can we write

$$i(t) = 1\,\mu\text{A}\cos(2\pi 1\,\text{kHz}\cdot t) - 4\pi\,\mu\text{A}\sin(2\pi 1\,\text{kHz}\cdot \text{t})$$

as a single sinusoid?

- Specifically, can we express it in the standard form

$$i(t) = I\cos(2\pi f t + \phi)$$

and, if so, what are I, f, and ϕ?

Solution

- Use the trig identity
  - $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

  to change $i(t) = I\cos(2\pi ft + \phi)$ to

$$i(t) = I \cdot \cos(\phi)\cos(2\pi ft) - I \cdot \sin(\phi)\sin(2\pi ft)$$

- Compare to

$$i(t) = 1\,\mu\text{A}\cos(2\pi 1\,\text{kHz} \cdot t) - 4\pi\,\mu\text{A}\sin(2\pi 1\,\text{kHz} \cdot \text{t})$$

- Conclude:
  - $f = 1\,\text{kHz}$ - no change in frequency!
  - $I \cdot \cos(\phi) = 1\,\mu\text{A}$ and $I \cdot \sin(\phi) = 4\pi\,\mu\text{A}$.

Solution

- We still must find I and ϕ from
  - $I \cdot \cos(\phi) = 1\,\mu\text{A}$ and $I \cdot \sin(\phi) = 4\pi\,\mu\text{A}$.
- We can find I from

$$\begin{array}{ccccc} I^2 \cdot \cos^2(\phi) & + & I^2 \cdot \sin^2(\phi) & = & I^2 \\ (1\,\mu\text{A})^2 & + & (4\pi\,\mu\text{A})^2 & \approx & (12.6\,\mu\text{A})^2 \end{array}$$

- Thus, $I = 12.6\,\mu\text{A}$.
- Also,

$$\frac{I \cdot \sin(\phi)}{I \cdot \cos(\phi)} = \tan(\phi) = \frac{4\pi}{1}.$$

- Hence, $\phi \approx 0.47 \cdot \pi \approx 85^o$.
- And, $i(t) \approx 12.6\,\mu\text{A} \cos(2\pi 1\,\text{kHz} \cdot t + 0.47 \cdot \pi)$.

Exercise

- Express

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2)$$

  in the form $A \cdot \cos(2\pi f t + \phi)$.
- Answer: $x(t) \approx 5 \cos(2\pi f t + 53^o)$

Solution to Exercise

- Express
$$x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi/2)$$
in the form $A \cdot \cos(2\pi ft + \phi)$.
- **Solution:** Use trig identity $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ on second term.
- This leads to
$$\begin{aligned} x(t) &= 3 \cdot \cos(2\pi ft) + \\ &\quad 4 \cdot \cos(2\pi ft)\cos(\pi/2) - 4 \cdot \sin(2\pi ft)\sin(\pi/2) \\ &= 3 \cdot \cos(2\pi ft) - 4 \cdot \sin(2\pi ft). \end{aligned}$$
- Compare to what we want:
$$\begin{aligned} x(t) &= A \cdot \cos(2\pi ft + \phi) \\ &= A \cdot \cos(\phi)\cos(2\pi ft) - A \cdot \sin(\phi)\sin(2\pi ft) \end{aligned}$$

Solution cont'd

- We can conclude that A and ϕ must satisfy

$$A \cdot \cos(\phi) = 3 \text{ and } A \cdot \sin(\phi) = 4.$$

- We can find A from

$$\begin{array}{ccccc} A^2 \cdot \cos^2(\phi) & + & A^2 \cdot \sin^2(\phi) & = & A^2 \\ 9 & + & 16 & = & 25 \end{array}$$

- Thus, $A = 5$.
- Also,

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{4}{3}.$$

- Hence, $\phi \approx 53^o$ ($\frac{53}{180}\pi$).
- And, $x(t) = 5\cos(2\pi ft + 53^o)$.

Summary

- Adding sinusoids of the same frequency is a problem that is frequently encountered in Electrical Engineering.
  - We noticed that the frequency of the sum of sinusoids is the same as the frequency of the sinusoids that we added.
- Such problems can be solved using trigonometric identities.
  - but, that is very tedious.
- We will see that sums of sinusoids are much easier to compute using complex algebra.