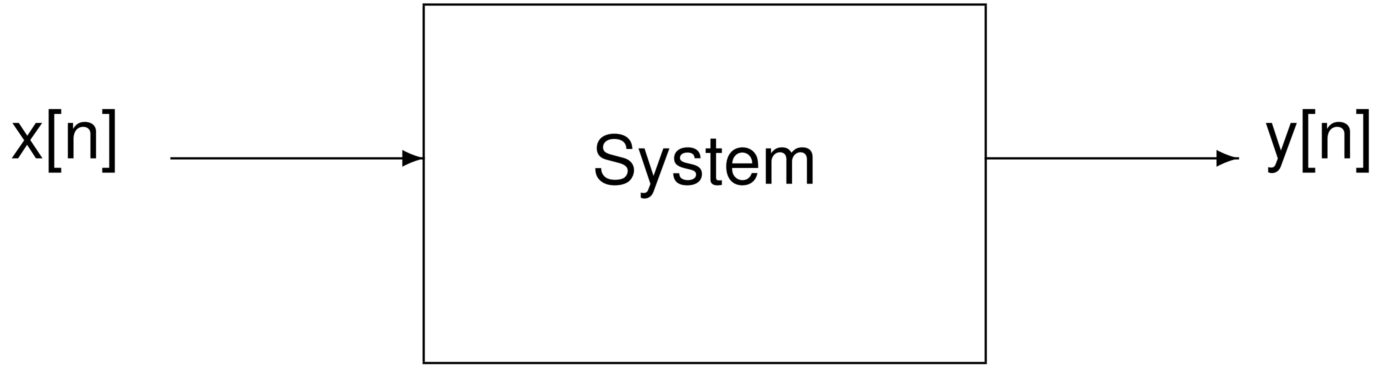
Lecture: Introduction to Systems and FIR filters

Systems

- A system is used to process an input signal $x[n]$ and produce the ouput signal $y[n]$.
  - We focus on discrete-time signals and systems;
  - a correspoding theory exists for continuous-time signals and systems.
- Many different systems:
  - Filters: remove undesired signal components,
  - Modulators and demodulators,
  - Detectors.



Representative Examples

- The following are examples of systems:
  - **Squarer:** $y[n] = (x[n])^2$;
  - **Modulator:** $y[n] = x[n] \cdot \cos(2\pi f_d n)$;
  - **Averager:** $y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$;
  - **FIR Filter:** $y[n] = \sum_{k=0}^{M} b_k x[n-k]$
- In MATLAB, systems are generally modeled as functions with $x[n]$ as the first input argument and $y[n]$ as the output argument.
  - **Example:** first two lines of function implementing a squarer.

```
function yy = squarer(xx)
% squarer - output signal is the square of the input signal
```

Squarer

- System relationship between input and output signals:

$$y[n] = (x[n])^2.$$

- **Example:** Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
  - **Notation:** $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$ means
    $x[0] = 1$, $x[1] = 2$, ..., $x[6] = 1$;
    all other $x[n] = 0$.
- Output signal: $y[n] = \{1, 4, 9, 16, 9, 4, 1\}$.

Modulator

- System relationship between input and output signals:

$$y[n] = (x[n]) \cdot \cos(2\pi f_d n);$$

  where the modulator frequency f_d is a *parameter* of the system.
- **Example:**
  - Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
  - assume $f_d = 0.5$, i.e., $\cos(2\pi f_d n) = \{\ldots, 1, -1, 1, -1, \ldots\}$.
- Output signal: $y[n] = \{1, -2, 3, -4, 3, -2, 1\}$.

Averager

- System relationship between input and output signals:

$$\begin{aligned} y[n] &= \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \\ &= \frac{1}{M}\cdot(x[n]+x[n-1]+\ldots+x[n-(M-1)]) \\ &= \sum_{k=0}^{M-1}\frac{1}{M}\cdot x[n-k]. \end{aligned}$$

- This system computes the *sliding average* over the M most recent samples.
- **Example:** Input signal: $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$
- For computing the output signal, a table is very useful.
  - synthetic multiplication table.

3-Point Averager ($M = 3$)

n	-1	0	1	2	3	4	5	6	7	8
$x[n]$	0	1	2	3	4	3	2	1	0	0
$\frac{1}{M} \cdot x[n]$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0
$+\frac{1}{M} \cdot x[n-1]$	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$+\frac{1}{M} \cdot x[n-2]$	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$
$y[n]$	0	$\frac{1}{3}$	1	2	3	$\frac{10}{3}$	3	2	1	$\frac{1}{3}$

- $y[n] = \{\frac{1}{3}, 1, 2, 3, \frac{10}{3}, 3, 2, 1, \frac{1}{3}\}$

General FIR Filter

- The M-point averager is a special case of the general FIR filter.
  - FIR stands for Finite Impulse Response; we will see what this means later.
- The system relationship between the input $x[n]$ and the output $y[n]$ is given by

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k].$$

General FIR Filter

- System relationship:

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k].$$

- The filter coefficients b_k determine the characteristics of the filter.
  - Much more on the relationship between the filter coefficients b_k and the characteristics of the filter later.
- Clearly, with $b_k = \frac{1}{M}$ for $k = 0, 1, \ldots, M-1$ we obtain the M-point averager.
- Again, computation of the output signal can be done via a synthetic multiplication table.
  - **Example:** $x[n] = \{1, 2, 3, 4, 3, 2, 1\}$ and $b_k = \{1, -2, 1\}$.

FIR Filter ($b_k = \{1, -2, 1\}$)

n	-1	0	1	2	3	4	5	6	7	8
$x[n]$	0	1	2	3	4	3	2	1	0	0
$1 \cdot x[n]$	0	1	2	3	4	3	2	1	0	0
$-2 \cdot x[n-1]$	0	0	-2	-4	-6	-8	-6	-4	-2	0
$+1 \cdot x[n-2]$	0	0	0	1	2	3	4	3	2	1
$y[n]$	0	1	0	0	0	-2	0	0	0	1

- $y[n] = \{1, 0, 0, 0, -2, 0, 0, 0, 1\}$
- Note that the output signal $y[n]$ is longer than the input signal $x[n]$.
- Note, synthetic multiplication works only for short, finite-duration signal.

Exercise

1. Find the output signal $y[n]$ for an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

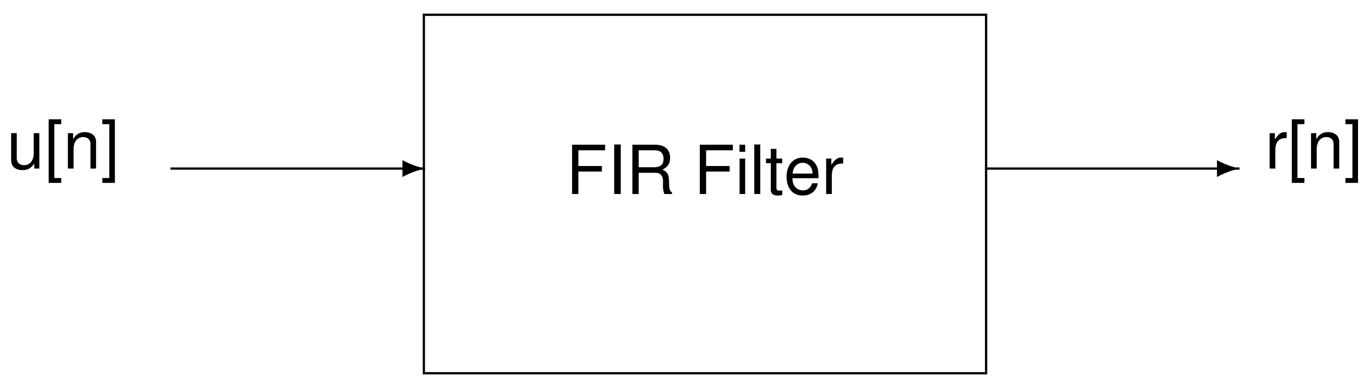
with filter coefficients $b_k = \{1, -1, 2\}$ when the input signal is $x[n] = \{1, 2, 4, 2, 4, 2, 1\}$.

Unit Step Sequence and Unit Step Response

- The signal with samples

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0 \end{cases}$$

  is called the unit-step sequence or unit-step signal.
- The output of an FIR filter when the input is the unit-step signal ($x[n] = u[n]$) is called the unit-step response $r[n]$.



Unit-Step Response of the 3-Point Averager

- Input signal: $x[n] = u[n]$.
- Output signal: $r[n] = \frac{1}{3}\sum_{k=0}^{2} u[n-k]$.

n	-1	0	1	2	3	...
$u[n]$	0	1	1	1	1	...
$\frac{1}{3}u[n]$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	...
$+\frac{1}{3}u[n-1]$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	...
$+\frac{1}{3}u[n-2]$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	...
$r[n]$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	1	...

Unit-Impulse Sequence and Unit-Impulse Response

- The signal with samples

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0 \end{cases}$$

  is called the unit-impulse sequence or unit-impulse signal.
- The output of an FIR filter when the input is the unit-impulse signal ($x[n] = \delta[n]$) is called the unit-impulse response, denoted $h[n]$.
- Typically, we will simply call the above signals simply impulse signal and impulse response.
- We will see that the impulse-response captures all characteristics of a FIR filter.
  - This implies that impulse response is a very important concept!

Unit-Impulse Response of a FIR Filter

- Input signal: $x[n] = \delta[n]$.
- Output signal: $h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$.

n	-1	0	1	2	3	...	M
$\delta[n]$	0	1	0	0	0	...	0
$b_0 \cdot \delta[n]$	0	b_0	0	0	0	...	0
$+b_1 \cdot \delta[n-1]$	0	0	b_1	0	0	...	0
$+b_2 \cdot \delta[n-2]$	0	0	0	b_2	0	...	0
$\vdots$				$\vdots$			
$+b_M \cdot \delta[n-M]$	0	0	0	0	0	...	b_M
$h[n]$	0	b_0	b_1	b_2	b_3	...	b_M

Important Insights

- For an FIR filter, the impulse response equals the sequence of filter coefficients:

$$h[n] = \begin{cases} b_n & \text{for } n = 0, 1, \ldots, M \\ 0 & \text{else.} \end{cases}$$

- Because of this relationship, the system relationship for an FIR filter can also be written as

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} b_k x[n-k] \\ &= \sum_{k=0}^{M} h[k] x[n-k] \\ &= \sum_{-\infty}^{\infty} h[k] x[n-k]. \end{aligned}$$

- The operation $y[n] = h[n] * x[n] = \sum_{-\infty}^{\infty} h[k] x[n-k]$ is called convolution; it is a **very, very** important operation.

Exercise

1. Find the impulse response $h[n]$ for the FIR filter with difference equation

$$y[n] = 2 \cdot x[n] + x[n-1] - 3 \cdot x[n-3].$$

2. Compute the output signal, when the input signal is $x[n] = u[n]$.
3. Compute the output signal, when the input signal is $x[n] = \exp(-\alpha n) \cdot u[n]$.