Fourier Series

- We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.
- One can turn this statement around and arrive at a very important result:

  Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.

- The resulting sum is called the Fourier Series of the signal.
- Put differently, a periodic signal can always be written in the form

$$\begin{aligned} x(t) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i f_0 t + \phi_i) \\ &= X_0 + \sum_{i=1}^{N} X_i e^{j2\pi i f_0 t} + X_i^* e^{-j2\pi i f_0 t} \end{aligned}$$

with $X_0 = A_0$ and $X_i = \frac{A_i}{2} e^{j\phi_i}$.

Fourier Series

- For a periodic signal the complex amplitudes X_i can be computed using a (relatively) simple formula.
- Specifically, for a periodic signal $x(t)$ with fundamental period T_0 the complex amplitudes X_i are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi it/T_0} dt.$$

  - Note that the integral above can be evaluated over any interval of length T_0.

Example: Square Wave

- A square wave signal is periodic and between $t - 0$ and $t = T_0$ it equals

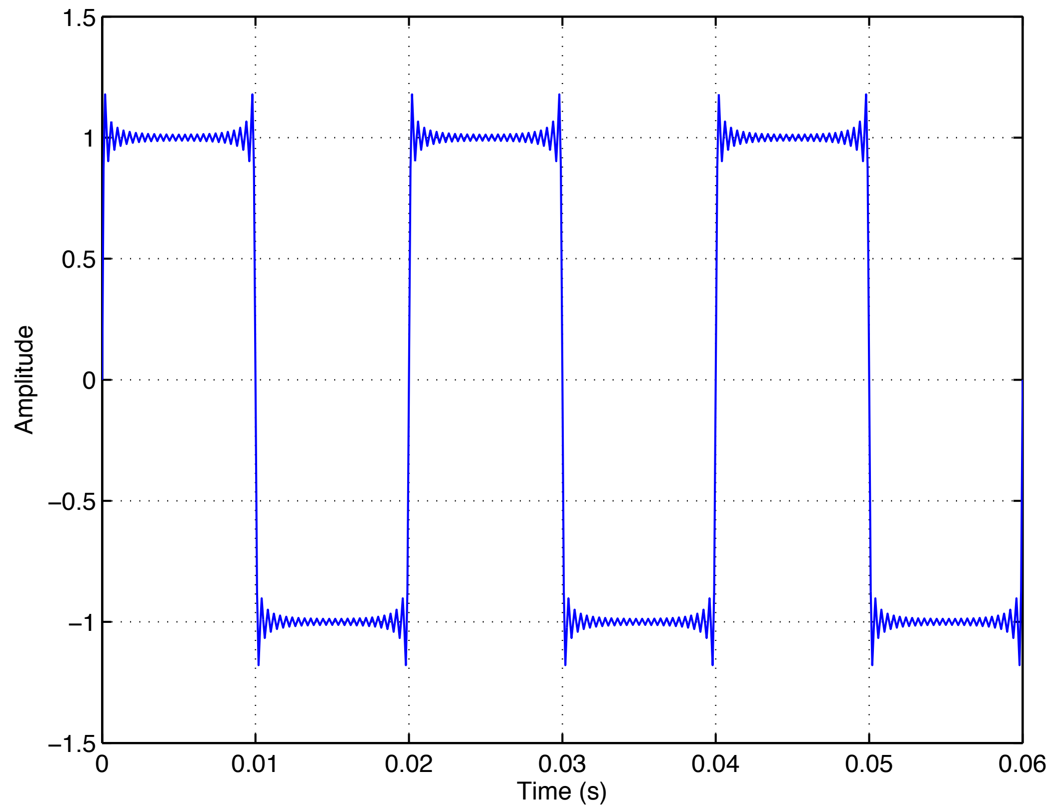
$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

- From the Fourier Series expansion it follows that $x(t)$ can be written as

$$x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$

25-Term Approximation to Square Wave

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$



Limitations of Sum-of-Sinusoid Signals

- So far, we have considered only signals that can be written as a sum of sinusoids.

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- For such signals, we are able to compute the spectrum.
- Note, that signals of this form
  - are assumed to last forever, i.e., for $-\infty < t < \infty$,
  - and their spectrum never changes.
- While such signals are important and useful conceptually, they don't describe real-world signals accurately.
- Real-world signals
  - are of finite duration,
  - their spectrum changes over time.

Musical Notation

- Musical notation ("sheet music") provides a way to represent real-world signals: a piece of music.
- As you know, sheet music
  - places notes on a scale to reflect the *frequency* of the tone to be played,
  - uses differently shaped note symbols to indicate the *duration* of each tone,
  - provides the order in which notes are to be played.
- In summary, musical notation captures how the spectrum of the music-signal changes over time.
- We cannot write signals whose spectrum changes with time as a sum of sinusoids.
  - A *static* spectrum is insufficient to describe such signals.
- Alternative: time-frequency spectrum

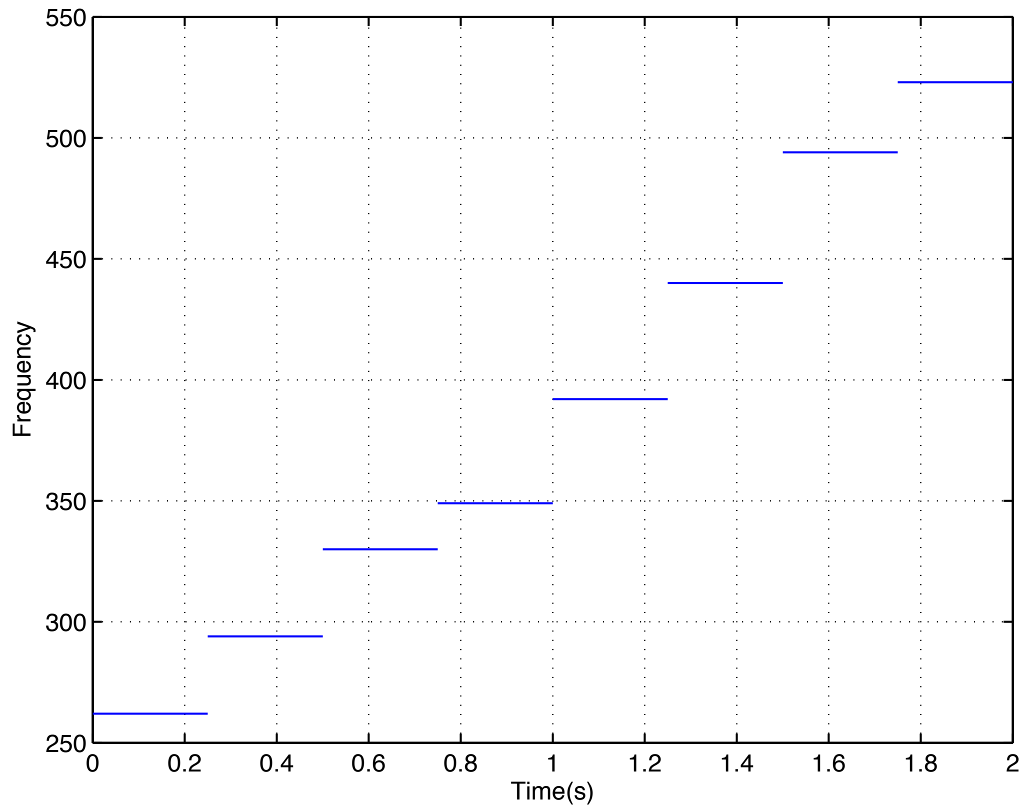
Example: Musical Scale

Note	C	D	E	F	G	A	B	C
Frequency (Hz)	262	294	330	349	392	440	494	523

Table: Musical Notes and their Frequencies

Example: Musical Scale

- If we play each of the notes for 250 ms, then the resulting signal can be summarized in the time-frequency spectrum below.

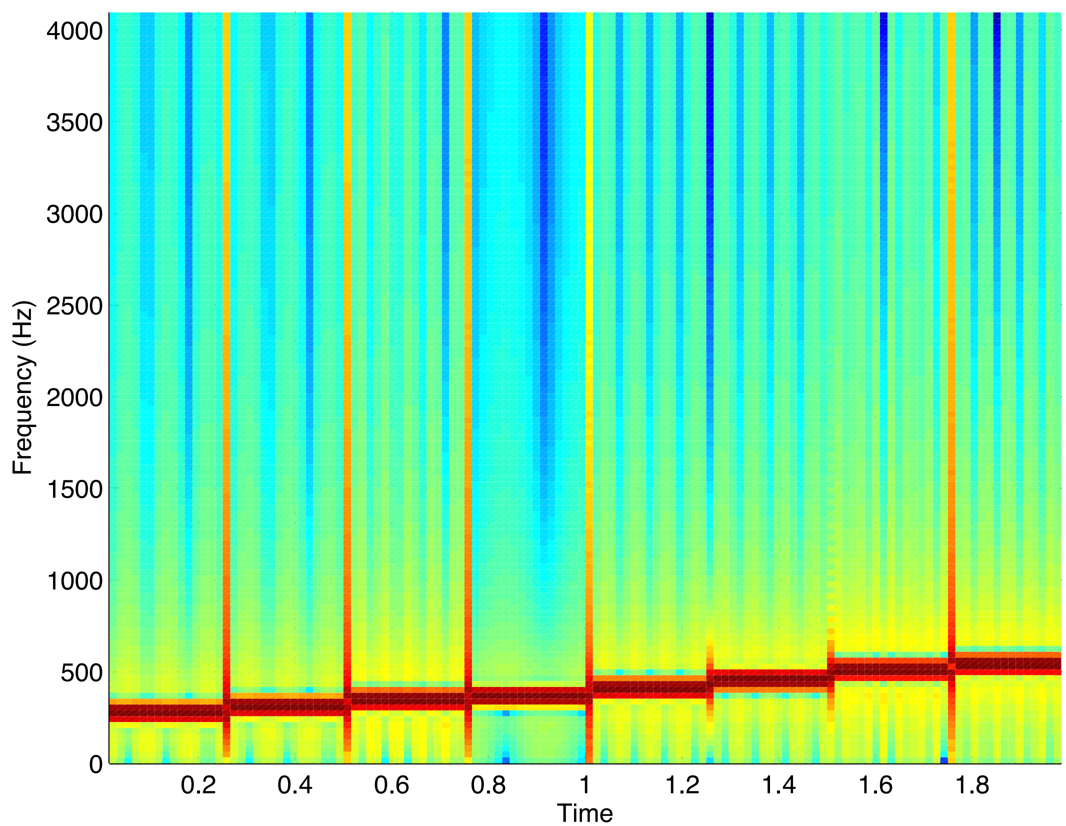


MATLAB Spectrogram Function

- MATLAB has a function `spectrogram` that can be used to compute the time-frequency spectrum for a given signal.
  - The resulting plots are similar to the one for the musical scale on the previous slide.
- Typically, you invoke this function as
  ```
  spectrogram( xx, 256, 128, 256,
  fs,'yaxis'),
  ```
  where `xx` is the signal to be analyzed and `fs` is the sampling frequency.
- The spectrogram for the musical scale is shown on the next slide.

Spectrogram: Musical Scale

- The color indicates the magnitude of the spectrum at a given time and frequency.



Chirp Signals

- **Objective:** construct a signal such that its frequency increases with time.
- **Starting Point:** A sinusoidal signal has the form:

$$x(t) = A\cos(2\pi f_0 t + \phi).$$

- We can consider the argument of the cos as a time-varying phase function

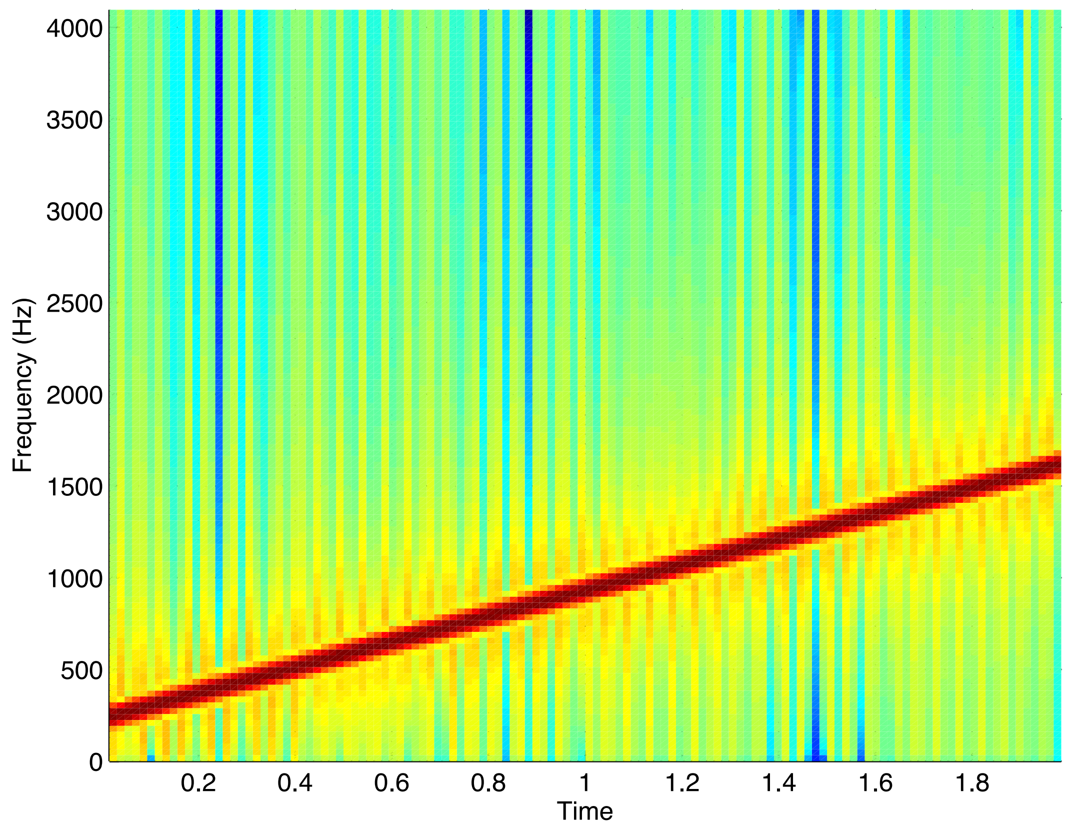
$$\Psi(t) = 2\pi f_0 t + \phi.$$

- **Question:** What happens when we allow more general functions for $\Psi(t)$?
  - For example, let

$$\Psi(t) = 700\pi t^2 + 440\pi t + \phi.$$

Spectrogram: $\cos(\Psi(t))$

- **Question:** How is he time-frequency spectrum related to $\Psi(t)$?



Instantaneous Frequency

- For a regular sinusoid, $\Psi(t) = 2\pi f_0 t + \phi$ and the frequency equals f_0.
- This suggests as a possible relationship between $\Psi(t)$ and f_0

$$f_0 = \frac{1}{2\pi} \frac{d}{dt} \Psi(t).$$

- If the above derivative is not a constant, it is called the instantaneous frequency of the signal, $f_i(t)$.
- **Example:** For $\Psi(t) = 700\pi t^2 + 440\pi t + \phi$ we find

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} (700\pi t^2 + 440\pi t + \phi) = 700t + 220.$$

  - This describes precisely the red line in the spectrogram on the previous slide.

Constructing a Linear Chirp

- **Objective:** Construct a signal such that its frequency is initially f_1 and increases linear to f_2 after T seconds.
- **Solution:** The above suggests that

$$f_i(t) = \frac{f_2 - f_1}{T} t + f_1.$$

- Consequently, the phase function $\Psi(t)$ must be

$$\Psi(t) = 2\pi \frac{f_2 - f_1}{2T} t^2 + 2\pi f_1 t + \phi$$

  - Note that ϕ has no influence on the spectrum; it is usually set to 0.

Constructing a Linear Chirp

- **Example:** Construct a linear chirp such that the frequency decreases from 1000 Hz to 200 Hz in 2 seconds.
- The desired signal must be

$$x(t) = \cos(-2\pi 200 t^2 + 2\pi 1000 t).$$

Exercise

- Construct a linear chirp such that the frequency increases from 50 Hz to 200 Hz in 3 seconds.
- Sketch the time-frequency spectrum of the following signal

$$x(t) = \cos(2\pi 500 t + 100 \cos(2\pi 2 t))$$