Lecture: Sums of Sinusoids (of different frequency)
Introduction

- To this point we have focused on sinusoids of identical frequency $f$

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi ft + \phi_i).$$

- Note that the frequency $f$ does not have a subscript $i$!
- Showed (via phasor addition rule) that the above sum can always be written as a single sinusoid of frequency $f$. 
Introduction

- We will consider sums of sinusoids of different frequencies:

\[ x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

- Note the subscript on the frequencies \( f_i \)!
- This apparently minor difference has dramatic consequences.
Sum of Two Sinusoids

\[ x(t) = \frac{4}{\pi} \cos(2\pi ft - \pi/2) + \frac{4}{3\pi} \cos(2\pi 3ft - \pi/2) \]
Sum of 25 Sinusoids

\[ x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2) \]
MATLAB: Sum of 25 Sinusoids

```matlab
f = 50;
fs = 200*f;

%% generate signals
% 5 cycles with 50 samples per cycle
tt = 0 : 1/fs : 3/f;

xx = zeros(size(tt));
for kk = 1:25
    xx = xx + 4/((2*kk-1)*pi)*cos(2*pi*(2*kk-1)*f*tt - pi/2);
end
```
MATLAB: Sum of 25 Sinusoids

The `for` loop can be replaced by:

```matlab
kk = (1:25);
xx = 4./((2*kk-1)*pi) * cos(2*pi*(2*kk'-1)*f*tt - pi/2);
```
Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusoids in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids
    - of frequencies \((2n - 1) \cdot f\), and
    - amplitudes \(\frac{4}{(2n - 1)\pi}\).
    - (This is not obvious → Fourier Series).
Non-sinusoidal Signals as Sums of Sinusoids

- The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.

- **Basic idea:** list the “*ingredients*” of a signal by specifying
  - amplitudes and phases, as well as
  - frequencies of the sinusoids in the sum.
The Spectrum of a Sum of Sinusoids

- Begin with the sum of sinusoids introduced earlier

\[ x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i). \]

where we have broken out a possible constant term.

- The term \( A_0 \) can be thought of as corresponding to a sinusoid of frequency zero.

- Using the inverse Euler formula, we can replace the sinusoids by complex exponentials

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i} \).
The Spectrum of a Sum of Sinusoids (cont’d)

Starting with

\[ x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}. \]

where \( X_0 = A_0 \) and \( X_i = A_i e^{j\phi_i} \).

The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

\[ X(f) = \{(X_0, 0), \left(\frac{X_1}{2}, f_1\right), \left(\frac{X_1^*}{2}, -f_1\right), \ldots, \left(\frac{X_N}{2}, f_N\right), \left(\frac{X_N^*}{2}, -f_N\right)\} \]
Example

Consider the signal

\[ x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4). \]

Using the inverse Euler relationship

\[ x(t) = 3 + \frac{5}{2} e^{-j\pi/2} \exp(j2\pi 10t) + \frac{5}{2} e^{j\pi/2} \exp(-j2\pi 10t) \]
\[ + \frac{7}{2} e^{j\pi/4} \exp(j2\pi 25t) + \frac{7}{2} e^{-j\pi/4} \exp(-j2\pi 25t). \]

Hence,

\[ X(f) = \{(3, 0), \left(\frac{5}{2} e^{-j\pi/2}, 10\right), \left(\frac{5}{2} e^{j\pi/2}, -10\right), \left(\frac{7}{2} e^{j\pi/4}, 25\right), \left(\frac{7}{2} e^{-j\pi/4}, -25\right)\} \]
Exercise

Find the spectrum of the signal:

\[ x(t) = 6 + 4 \cos(10\pi t + \pi/3) + 5 \cos(20\pi t - \pi/7). \]
Time-domain and Frequency-domain

- Signals are *naturally* observed in the time-domain.
- A signal can be illustrated in the time-domain by plotting it as a function of time.
- The frequency-domain provides an alternative perspective of the signal based on sinusoids:
  - Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
  - The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
  - Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of these parameters.
    - Actually, we list pairs of complex amplitudes ($Ae^{j\phi}$) and frequencies $f$ and refer to this list as $X(f)$. 
It is possible (but not necessarily easy) to find $X(f)$ from $x(t)$: this is called Fourier or spectrum analysis.

Similarly, one can construct $x(t)$ from the spectrum $X(f)$: this is called Fourier synthesis.

Notation: $x(t) \leftrightarrow X(f)$.

Example (from earlier):

**Time-domain:** signal

$$x(t) = 3 + 5 \cos(20\pi t - \pi/2) + 7 \cos(50\pi t + \pi/4).$$

**Frequency Domain:** spectrum

$$X(f) = \{(3, 0), \left(\frac52 e^{-j\pi/2}, 10\right), \left(\frac52 e^{j\pi/2}, -10\right), \left(\frac72 e^{j\pi/4}, 25\right), \left(\frac72 e^{-j\pi/4}, -25\right)\}.$$
Plotting a Spectrum

- To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.
- Sometimes the phase is plotted versus frequency as well.
Why Bother with the Frequency-Domain?

- In many applications, the frequency contents of a signal is very important.
  - For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
  - Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.

- Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).

- We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.
  - Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.
Example: Original signal
Example: Corrupted signal
Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- **Ingredients** are given by the spectrum

\[ X(f) = \{ (X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N) \} \]

Each pair indicates one complex exponential component by listing its frequency and complex amplitude.

- **Instructions** for combining the ingredients and producing the (time-domain) signal:

\[ x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t). \]

- Always simplify the expression you obtain!
Example

<table>
<thead>
<tr>
<th>Problem: Find the signal $x(t)$ corresponding to $X(f) = {(3, 0), (\frac{5}{2}e^{-j\pi/2}, 10), (\frac{5}{2}e^{j\pi/2}, -10), (\frac{7}{2}e^{j\pi/4}, 25), (\frac{7}{2}e^{-j\pi/4}, -25)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution: $x(t) = 3 + \frac{5}{2}e^{-j\pi/2}e^{j2\pi10t} + \frac{5}{2}e^{j\pi/2}e^{-j2\pi10t} + \frac{7}{2}e^{j\pi/4}e^{j2\pi25t} + \frac{7}{2}e^{-j\pi/4}e^{-j2\pi25t}$</td>
</tr>
<tr>
<td>Which simplifies to: $x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4)$.</td>
</tr>
</tbody>
</table>
Exercise

Find the signal with the spectrum:

\[ X(f) = \{(5, 0), \quad (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \]
\[ (\frac{5}{2}e^{j\pi/4}, 15), (\frac{5}{2}e^{-j\pi/4}, -15) \]