Lecture: Sums of Sinusoids (of different frequency)

Introduction

- To this point we have focused on sinusoids of identical frequency f

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i).$$

  - Note that the frequency f does not have a subscript i!
- Showed (via phasor addition rule) that the above sum can always be written as a single sinusoid of frequency f.

Introduction

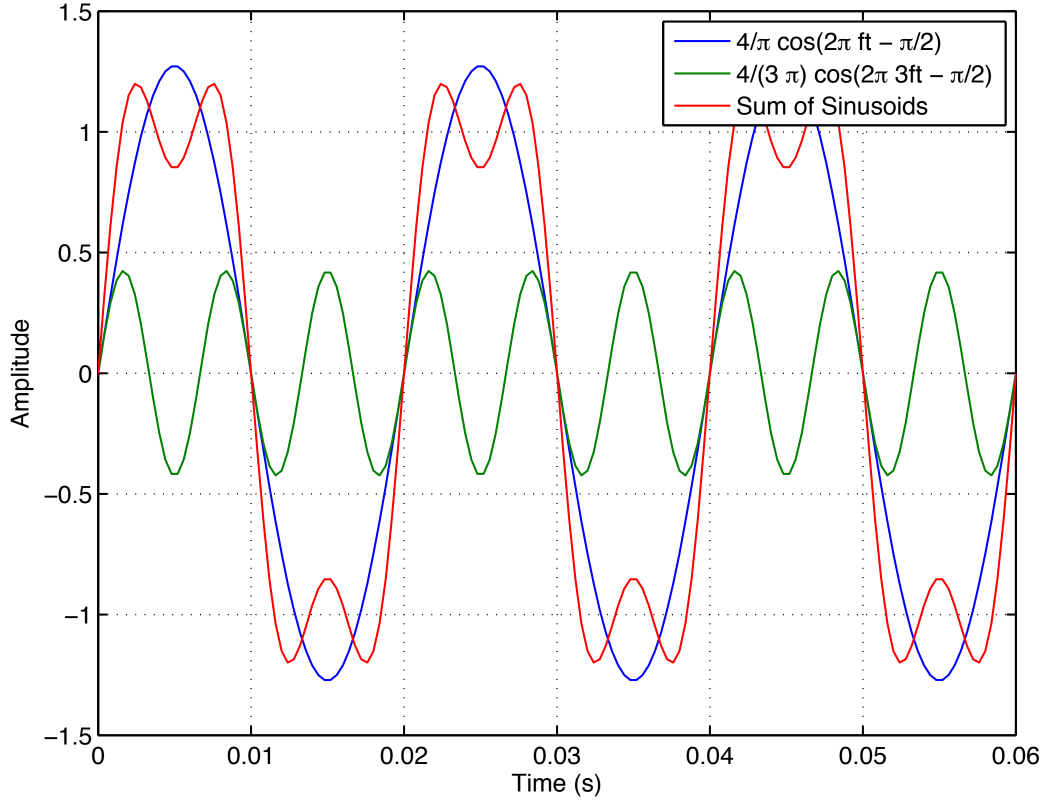
- We will consider sums of sinusoids of different frequencies:

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

  - Note the subscript on the frequencies f_i!
  - This apparently minor difference has dramatic consequences.

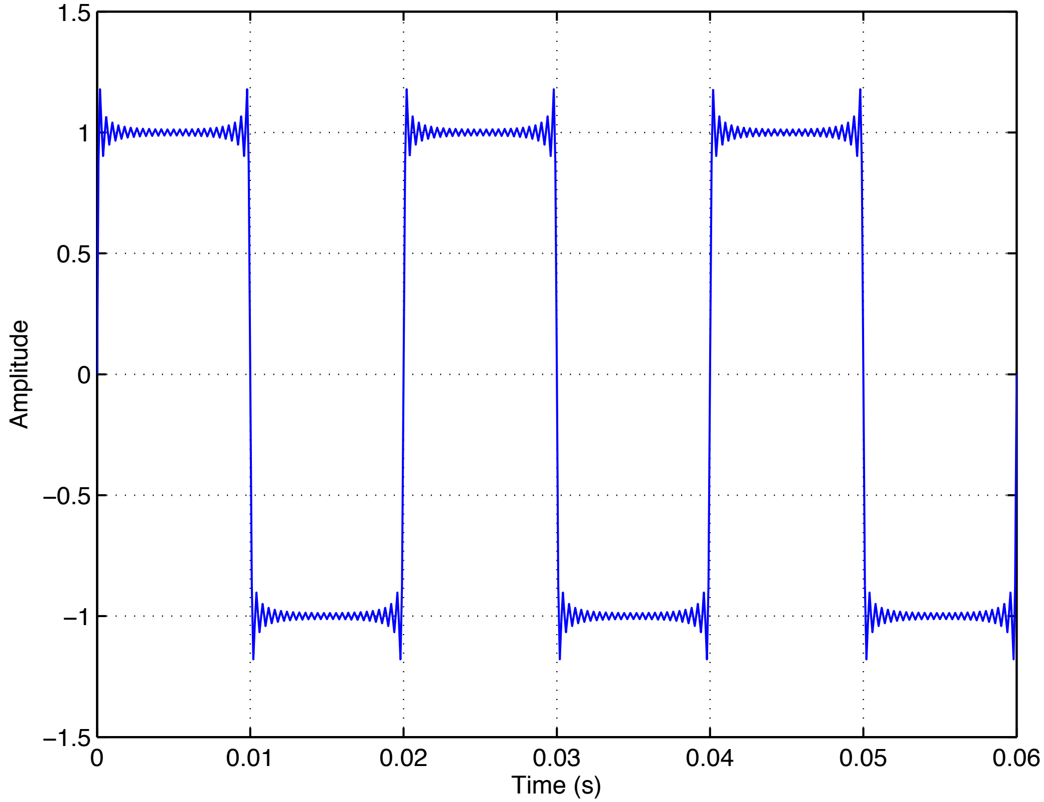
Sum of Two Sinusoids

$$x(t) = \frac{4}{\pi}\cos(2\pi ft - \pi/2) + \frac{4}{3\pi}\cos(2\pi 3ft - \pi/2)$$



Sum of 25 Sinusoids

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$



MATLAB: Sum of 25 Sinusoids

```
f   = 50;
fs  = 200*f;

%% generate signals
% 5 cycles with 50 samples per cycle
tt = 0 : 1/fs : 3/f;

xx = zeros(size(tt));
for kk = 1:25
    xx = xx + 4/((2*kk-1)*pi)*cos(2*pi*(2*kk-1)*f*tt - pi/2);
end
```

MATLAB: Sum of 25 Sinusoids

- The `for` loop can be replaced by:

```
kk = (1:25);
xx = 4./((2*kk-1)*pi) * cos(2*pi*(2*kk'-1)*f*tt - pi/2);
```

Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusods in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids
    - of frequencies $(2n-1) \cdot f$, and
    - amplitudes $\frac{4}{(2n-1)\pi}$.
    - (This is not obvious $\rightarrow$ Fourier Series).

Non-sinusoidal Signals as Sums of Sinusoids

- The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.
- **Basic idea:** list the *"ingredients"* of a signal by specifying
  - amplitudes and phases, as well as
  - frequencies of the sinusoids in the sum.

The Spectrum of a Sum of Sinusoids

- Begin with the sum of sinusoids introduced earlier

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

where we have broken out a possible constant term.
  - The term A_0 can be thought of as corresponding to a sinusoid of frequency zero.
- Using the *inverse Euler formula*, we can replace the sinusoids by complex exponentials

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

The Spectrum of a Sum of Sinusoids (cont'd)

- Starting with

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

- The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

$$X(f) = \{(X_0, 0), (\frac{X_1}{2}, f_1), (\frac{X_1^*}{2}, -f_1), \ldots, (\frac{X_N}{2}, f_N), (\frac{X_N^*}{2}, -f_N)\}$$

Example

- Consider the signal

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

- Using the inverse Euler relationship

$$\begin{aligned} x(t) = 3 \quad &+ \quad \tfrac{5}{2}e^{-j\pi/2}\exp(j2\pi 10t) \quad + \quad \tfrac{5}{2}e^{j\pi/2}\exp(-j2\pi 10t) \\ &+ \quad \tfrac{7}{2}e^{j\pi/4}\exp(j2\pi 25t) \quad + \quad \tfrac{7}{2}e^{-j\pi/4}\exp(-j2\pi 25t). \end{aligned}$$

- Hence,

$$\begin{aligned} X(f) = \{(3,0), \quad &(\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ &(\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\} \end{aligned}$$

Exercise

- Find the spectrum of the signal:

$$x(t) = 6 + 4\cos(10\pi t + \pi/3) + 5\cos(20\pi t - \pi/7).$$

Time-domain and Frequency-domain

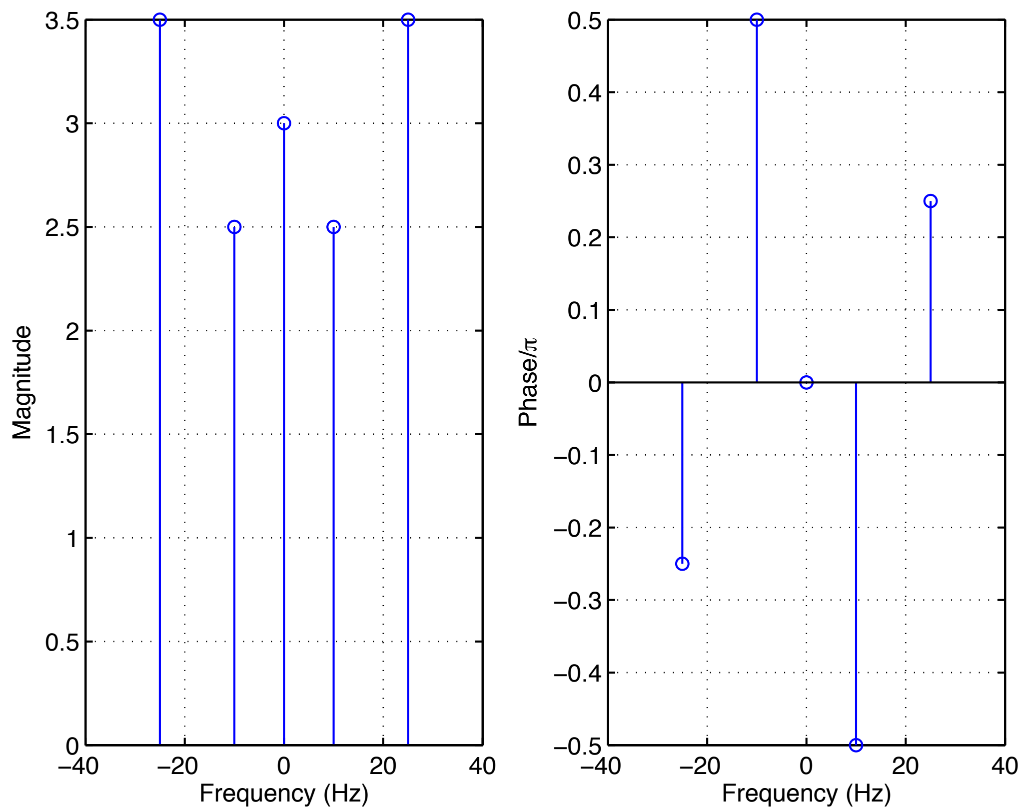
- Signals are *naturally* observed in the time-domain.
- A signal can be illustrated in the time-domain by plotting it as a function of time.
- The frequency-domain provides an alternative perspective of the signal based on sinusoids:
  - Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
  - The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
  - Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of theses parameters.
    - Actually, we list pairs of complex amplitudes ($Ae^{j\phi}$) and frequencies f and refer to this list as $X(f)$.

Time-domain and Frequency-domain

- It is possible (but not necessarily easy) to find $X(f)$ from $x(t)$: this is called Fourier or spectrum analysis.
- Similarly, one can construct $x(t)$ from the spectrum $X(f)$: this is called Fourier synthesis.
- Notation: $x(t) \leftrightarrow X(f)$.
- Example (from earlier):
  - **Time-domain:** signal
$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$
  - **Frequency Domain:** spectrum
$$X(f) = \{(3,0), \quad (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}$$

Plotting a Spectrum

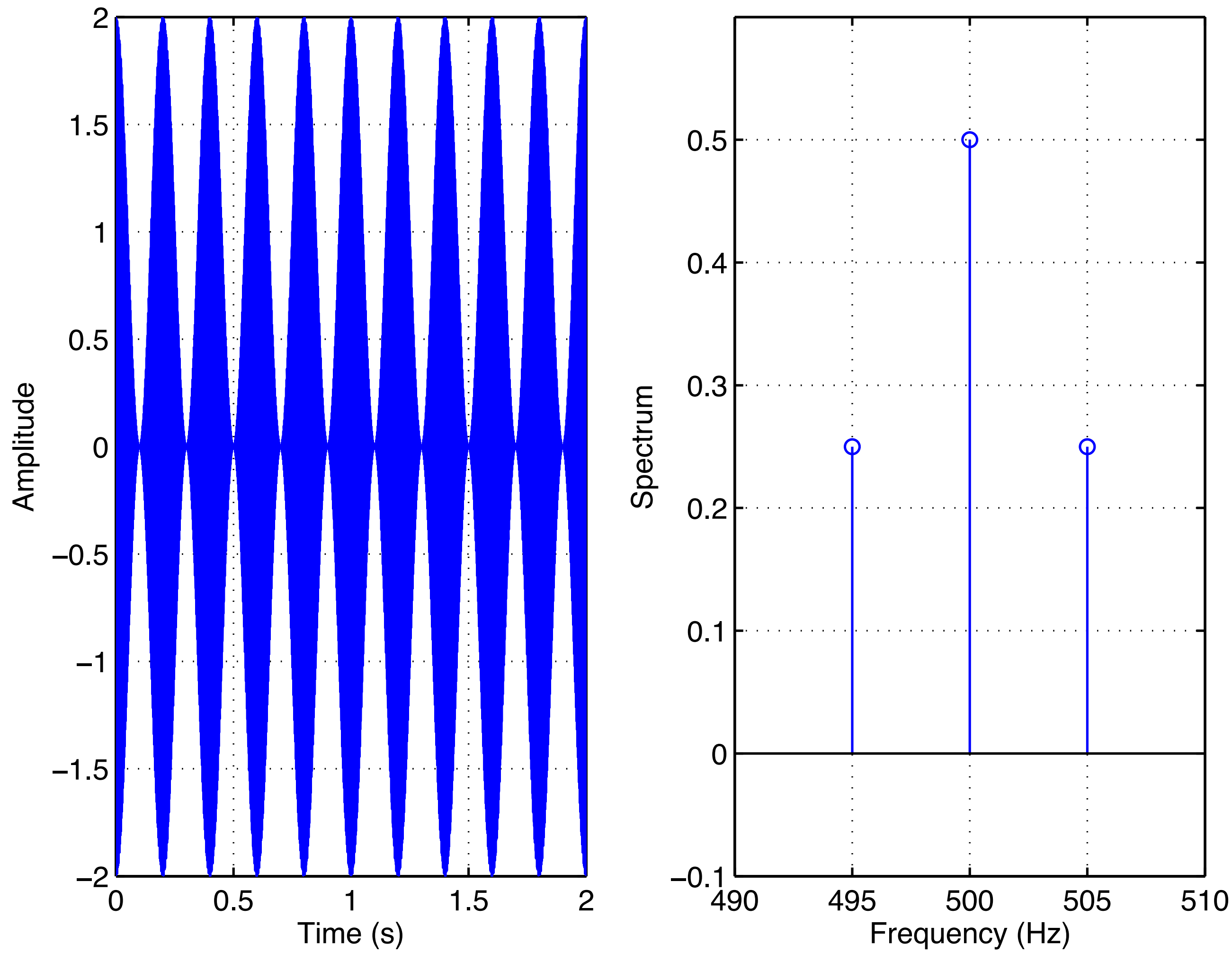
- To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.
- Sometimes the phase is plotted versus frequency as well.



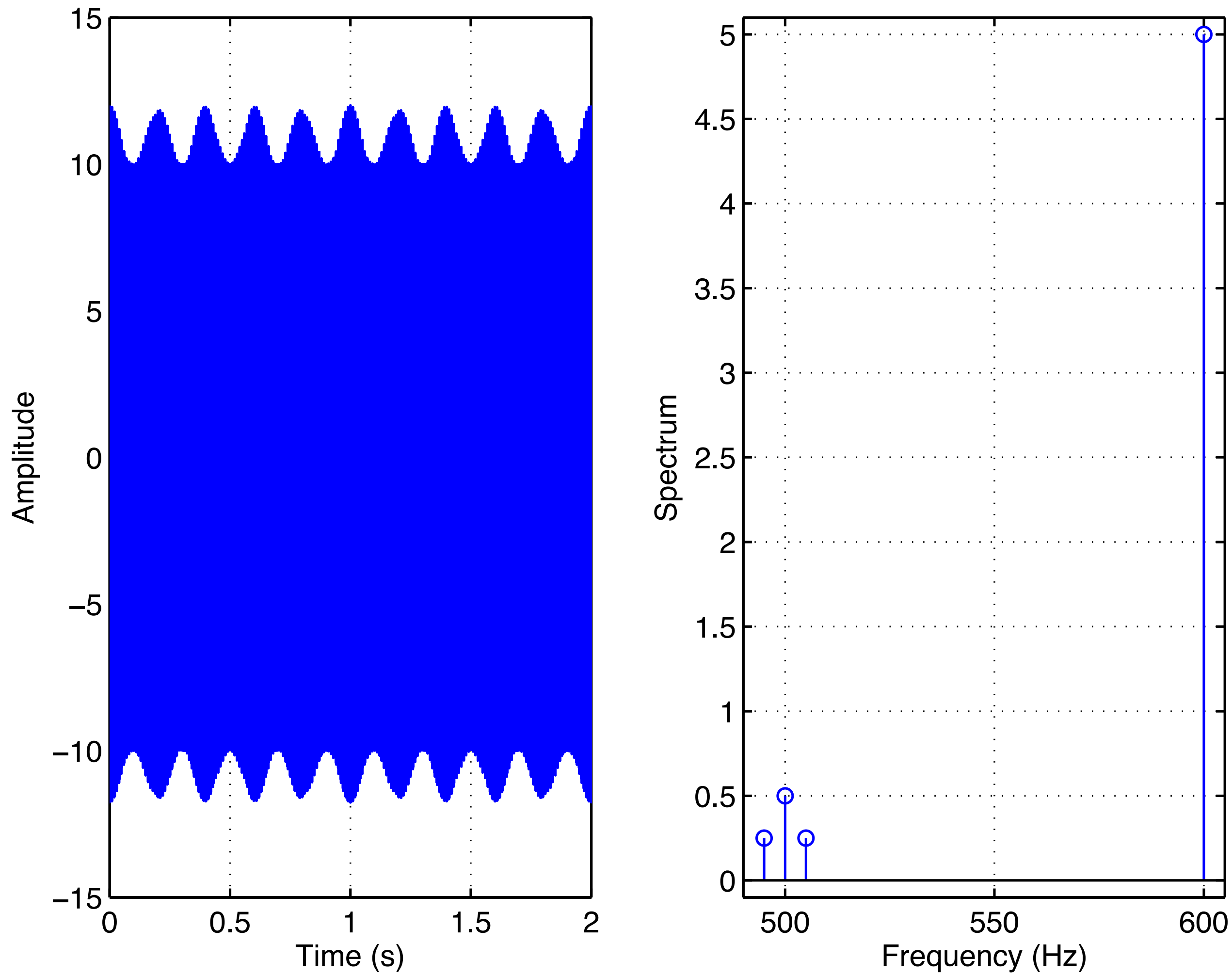
Why Bother with the Frequency-Domain?

- In many applications, the frequency contents of a signal is very important.
  - For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
  - Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.
- Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).
- We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.
  - Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.

Example: Original signal



Example: Corrupted signal



Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- *Ingredients* are given by the spectrum

$$X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\}$$

  Each pair indicates one complex exponential component by listing its frequency and complex amplitude.
- *Instructions* for combining the ingredients and producing the (time-domain) signal:

$$x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t).$$

- Always simplify the expression you obtain!

Example

- Problem: Find the signal $x(t)$ corresponding to

$$X(f) = \{(3,0), \quad (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}$$

- Solution:

$$x(t) = 3 \quad +\tfrac{5}{2}e^{-j\pi/2}e^{j2\pi 10t} + \tfrac{5}{2}e^{j\pi/2}e^{-j2\pi 10t} \\ +\tfrac{7}{2}e^{j\pi/4}e^{j2\pi 25t} + \tfrac{7}{2}e^{-j\pi/4}e^{-j2\pi 25t}$$

- Which simplifies to:

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

Exercise

- Find the signal with the spectrum:

$$\begin{aligned} X(f) = \{(5,0),\quad & (2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \\ & (\tfrac{5}{2}e^{j\pi/4}, 15), (\tfrac{5}{2}e^{-j\pi/4}, -15) \end{aligned}$$