ECE 201: Introduction to Signal Analysis

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Part I

Introduction

Lecture: Introduction

Learning Objectives

- Intro to Electrical Engineering via Digital Signal Processing.
- Develop initial understanding of Signals and Systems.
- Learn MATLAB
- Note: Math is not very hard - just algebra.

DSP

Digital: processing via computers and digital hardware
we will use PC’s.

Signal: Principally signals are just functions of time

- Entertainment/music
- Communications
- Medical, . . .

Processing: analysis and transformation of signals
we will use MATLAB

Outline of Topics

- Sinusoidal Signals
- Time and Frequency representation of signals
- Sampling
- Filtering

- MATLAB
    - Lectures
    - Labs
    - Homework

Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expresssed as sums of sinusoids (Fourier Theory)
- Bridge to frequency domain.
- Sinusoids are *special signals* for linear filters (eigenfunctions).

Time and Frequency

- Closely related via sinusoids.
- Provide two different perspectives on signals.
- Many operations are easier to understand in frequency domain.

Sampling

- Conversion from continuous time to discrete time.
- Required for Digital Signal Processing.
- Converts a signal to a sequence of numbers (samples).
- Straightforward operation
  - with a few *strange* effects.

Filtering

- A simple, but powerful, class of operations on signals.
- Filtering transforms an *input signal* into a more suitable *output signal*.
- Often best understood in frequency domain.

Relationship to other ECE Courses

- ECE 220/320: Signals and Systems
- ECE 280: Circuits
- ECE 421: Controls
- ECE 460: Communications
- ECE 410: DSP
- ECE 450: Robotics
- ECE 463: Digital Comms
- ECE 464: Filter Design

Part II

Sinusoids, Complex Numbers, and Complex Exponentials

Lecture: Introduction to Sinusoids

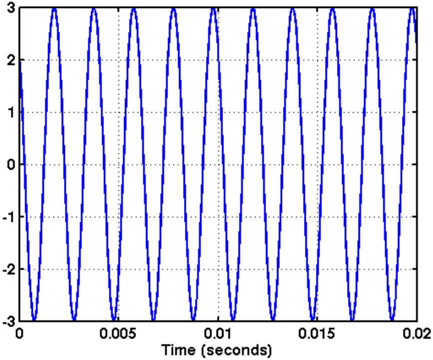
The Formula for Sinusoidal Signals

- The general formula for a sinusoidal signal is

$$x(t) = A \cdot \cos(2\pi f t + \phi).$$

- A, f, and ϕ are parameters that characterize the sinusoidal sinal.
  - A - Amplitude: determines the height of the sinusoid.
  - f - Frequency: determines the number of cycles per second.
  - ϕ - Phase: determines the location of the sinusoid.

Example



- The formula for this sinusoid is:

$$x(t) = 3 \cdot \cos(2\pi \cdot 500 \cdot t + \pi/4).$$

The Significance of Sinusoidal Signals

- Fundamental building blocks for describing arbitrary signals.
  - General signals can be expresssed as sums of sinusoids (Fourier Theory)
  - Provides bridge to frequency domain.
- Sinusoids are *special signals* for linear filters (eigenfunctions).
- Sinusoids occur naturally in many situations.
  - They are solutions of differential equations of the form

$$\frac{d^2x(t)}{dt^2} + ax(t) = 0.$$

- Much more on these points as we proceed.

Background: The cosine function

- The properties of sinusoidal signals stem from the properties of the cosine function:
  - Periodicity: $\cos(x + 2\pi) = \cos(x)$
  - Eveness: $\cos(-x) = \cos(x)$
  - Ones of cosine: $\cos(2\pi k) = 1$, for all integers k.
  - Minus ones of cosine: $\cos(\pi(2k + 1)) = -1$, for all integers k.
  - Zeros of cosine: $\cos(\frac{\pi}{2}(2k + 1)) = 0$, for all integers k.
  - Relationship to sine function: $\sin(x) = \cos(x - \pi/2)$ and $\cos(x) = \sin(x + \pi/2)$.

Amplitude

- The amplitude A is a *scaling factor*.
- It determines how large the signal is.
- Specifically, the sinusoid oscillates between $+A$ and $-A$.

Frequency and Period

- Sinusoids are periodic signals.
- The frequency f indicates how many times the sinusoid repeats per second.
- The duration of each cycle is called the period of the sinusoid.
  It is denoted by T.
- The relationship between frequency and period is

$$f = \frac{1}{T} \text{ and } T = \frac{1}{f}.$$

Phase and Delay

- The phase ϕ causes a sinusoid to be shifted sideways.
- A sinusoid with phase $\phi = 0$ has a maximum at $t = 0$.
- A sinusoid that has a maximum at $t = t_1$ can be written as

$$x(t) = A \cdot \cos(2\pi f(t - t_1)).$$

- Expanding the argument of the cosine leads to

$$x(t) = A \cdot \cos(2\pi ft - 2\pi ft_1).$$

- Comparing to the general formula for a sinusoid reveals

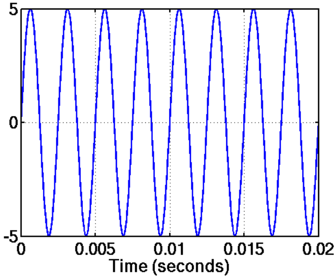
$$\phi = -2\pi ft_1 \text{ and } t_1 = \frac{-\phi}{2\pi f}.$$

Exercise

1. Plot the sinusoid

$$x(t) = 2\cos(2\pi \cdot 10 \cdot t + \pi/2)$$

   between $t = -0.1$ and $t = 0.2$.
2. Find the equation for the sinusoid in the following plot



Vectors and Matrices

- MATLAB is specialized to work with vectors and matrices.
- Most MATLAB commands take vectors or matrices as arguments and perform looping operations automatically.
- Creating vectors in MATLAB:

directly

```
x = [ 1, 2, 3 ];
```

using the increment (:) operator

```
x = 1:2:10;
```

produces a vector with elements $[1, 3, 5, 7, 9]$.

from external data For example, by reading a .wav file

```
[ x, fs] = wavread('music.wav');
```

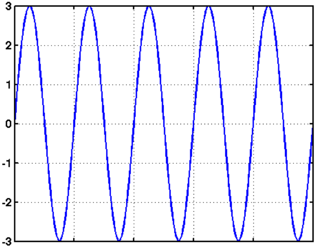
Plot a Sinusoid

```
f  = 500;    % frequency
fs = 50*f;   % sampling frequency

tt = 0 : 1/fs : 10/f; % time axis: 10 cycles
xx = 5*cos(2*pi*f*tt + pi/4);

plot(tt,xx);
grid
xlabel('Time (s)')
```

Lecture: Continuous-time and Discrete-Time Signals

Exercise

- The sinusoid below has frequency $f = 10$ Hz.
- Three of its maxima are at the the following locations $t_1 = -0.075$ s, $t_2 = 0.025$ s, $t_3 = 0.125$ s
- Use each of these three delays to compute a value for the phase ϕ.
- What is the relationship between the phase values you obtain?



Continuous-Time Signals

- So far, we have been refering to sinusoids of the form

$$x(t) = A \cdot \cos(2\pi ft + \phi).$$

- Here, the independent variable *t* is continuous, i.e., it takes on a continuum of values.
  - Signals that are functions of a continuous time variable *t* are called continuous-time signals or, sometimes, analog signals.

Sampling and Discrete-Time Signals

- MATLAB, and other digital processing systems, can not process continuous-time signals.
- Instead, MATLAB requires the continuous-time signal to be converted into a discrete-time signal.
- The conversion process is called sampling.
- To sample a continuous-time signal, we evaluate it at a discrete set of times $t_n = nT_s$, where
    - n is a integer,
    - T_s is called the sampling period (time between samples),
    - $f_s = 1/T_s$ is the sampling rate (samples per second).

Sampling and Discrete-Time Signals

- Sampling results in a sequence of samples

$$x(nT_s) = A \cdot \cos(2\pi f n T_s + \phi).$$

- Note that the independent variable is now *n*, not *t*.
- To emphasize that this is a discrete-time signal, we write

$$x[n] = A \cdot \cos(2\pi f n T_s + \phi).$$

- Sampling is a straightforward operation.
- But the sampling rate must be chosen with care!

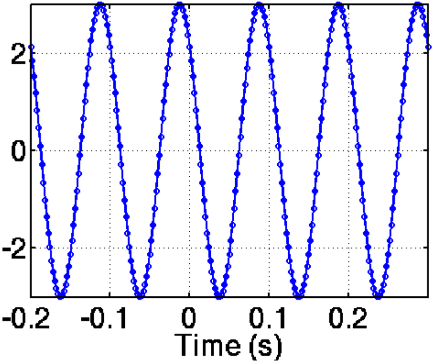
Plot a Sinusoid - Improved

```
% Plot sinusoidal signals

% Parameters of sinusoid:
A   = 3;      % Amplitude
f   = 10;     % Frequency
phi = pi/4;   % Phase;

% Parameters controlling plot:
SamplesPerCycle = 50;
CyclesToPlot    = 5;
StartTime       = -0.2;

fs      = f * SamplesPerCycle;
EndTime = StartTime + CyclesToPlot/f;
```

Plot a Sinusoid - Improved

```
tt = StartTime : 1/fs : EndTime; % Time axis
xx = A*cos(2*pi*f*tt + phi);

plot(tt,xx,'-o');
grid;
xlabel( 'Time (s)');
```

Resulting Plot



- The solid line indicates the continuous-time signal $x(t)$.
- The circles represent the samples that make up the discrete-time signal $x[n]$.

Deciphering the MATLAB code

- The code is written to plot a specified number of cycles (`CyclesToPlot`) with a given number of samples per cycle (`SamplesPerCycle`).
- This implies that the sampling rate f_s equals the product of `SamplesPerCycle` and `CyclesToPlot`.
- The duration of the signal follows from the specification of the number of cycles to plot: `CyclesToPlot / f`.
- With a given starting time (`StartTime`), the discrete set of time instances t_n is constructed by

  ```
  tt = StartTime : 1/fs : EndTime;
  ```
- The `cos` function can be called with a vector as its argument to compute all desired values; no loops!

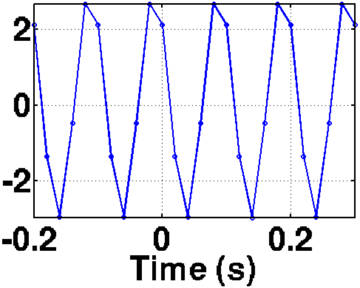
  ```
  xx = A*cos(2*pi*f*tt + phi);
  ```

Some Tips MATLAB programming

- Comment your code (comments start with %.)
- Use descriptive names for variables (`SamplesPerCycle`).
- Avoid loops!
- If the above MATLAB code is stored in a file, say `PlotSinusoid.m`, then it can be executed by typing `PlotSinusoid`.
  - Filename must end in `.m`
  - File must be in your working directory,
  - or more generally in your path.

Reducing the Sampling Rate

- What happens if we reduce the sampling rate?
  E.g., by setting `SamplesPerCycle = 5;`



- The sampling rate is not high enough to create an accurate plot.

Lecture: Introduction to Complex Numbers

Why Complex Numbers?

- Complex numbers are closely related to sinusoids.
- They eliminate the need for trigonometry
- and replace it with simple algebra.
    - Complex algebra is really simple - this is not an oxymoron.
- Complex numbers can be represented as vectors
    - Used to visualize the relationship between sinusoids.

An (unpleasant) Example

- **A typical problem:** Express
$$x(t) = 3 \cdot \cos(2\pi ft) + 4 \cdot \cos(2\pi ft + \pi/2)$$
in the form $A \cdot \cos(2\pi ft + \phi)$.
- **Solution:** Use trig identity
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ on second term.
- Leads to
$$\begin{aligned} x(t) &= 3 \cdot \cos(2\pi ft) + \\ &\quad 4 \cdot \cos(2\pi ft)\cos(\pi/2) - 4 \cdot \sin(2\pi ft)\sin(\pi/2) \\ &= 3 \cdot \cos(2\pi ft) - 4 \cdot \sin(2\pi ft). \end{aligned}$$
- Want:
$$\begin{aligned} x(t) &= A \cdot \cos(2\pi ft + \phi) \\ &= A \cdot \cos(\phi)\cos(2\pi ft) - A \cdot \sin(\phi)\sin(2\pi ft) \end{aligned}$$

More Unpleasantness ...

- We can conclude that A and ϕ must satisfy

$$A \cdot \cos(\phi) = 3 \text{ and } A \cdot \sin(\phi) = 4.$$

- We can find A from

$$\begin{array}{ccccc} A^2 \cdot \cos^2(\phi) & + & A^2 \cdot \sin^2(\phi) & = & A^2 \\ 9 & + & 16 & = & 25 \end{array}$$

- Thus, $A = 5$.
- Also,

$$\frac{\sin(\phi)}{\cos \phi} = \tan(\phi) = \frac{4}{3}.$$

- Hence, $\phi \approx 53^o$ $(\frac{53}{180}\pi)$.
- And, $x(t) = 5\cos(2\pi ft + 53^o)$.
- With complex numbers problems of this type are much easier.

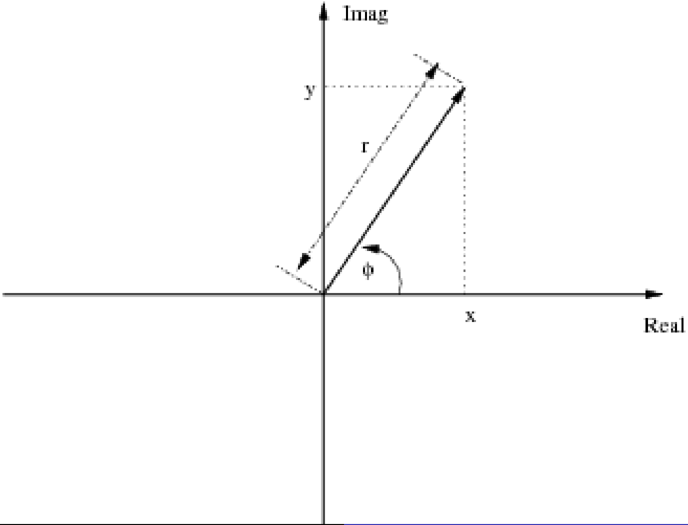
The Basics

- Complex unity: $j = \sqrt{-1}$.
- Complex numbers can be written as

$$z = x + j \cdot y.$$

  This is called the rectangular or cartesian form.
    - x is called the real part of z: $x = \text{Re}\{z\}$.
    - y is called the imaginary part of z: $y = \text{Im}\{z\}$.
- z can be thought of a vector in a two-dimensional plane.
    - Cordinates are x and y.
    - Coordinate system is called the complex plane.

Illustration



Euler's Formulas

- Euler's formula provides the connection between complex numbers and trigonometric functions.

$$e^{j\phi} = \cos(\phi) + j \cdot \sin(\phi).$$

- Euler's formula allows conversion between trigonometric functions and exponentials.
  - Exponentials have simple algebraic rules!
- Inverse Euler's formulas:

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

- These relationships are very important.

Polar Form

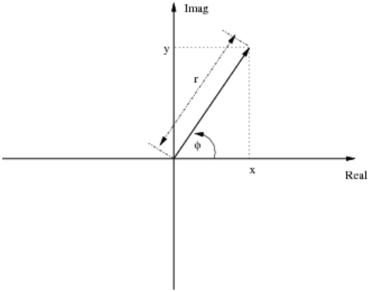
- Recall $z = x + j \cdot y$
- From the diagram it follows that

$$z = r\cos(\phi) + jr\sin(\phi).$$

- And by Euler's relationship, it follows that

$$\begin{aligned} z &= r \cdot (\cos(\phi) + j\sin(\phi)) \\ &= r \cdot e^{j\phi} \end{aligned}$$

- This is called the polar form.



Converting from Polar to Cartesian Form

- A complex number polar form $z = r \cdot e^{j\phi}$ is easily converted to cartesian form.

$$z = r\cos(\phi) + jr\sin(\phi).$$

- **Example:**

$$\begin{aligned} 4 \cdot e^{j\pi/3} &= 4\cos(\pi/3) + j4\sin(\pi/3) \\ &= 4\tfrac{1}{2} + j4\tfrac{\sqrt{3}}{2} \\ &= 2 + j2\sqrt{3}. \end{aligned}$$

Converting from Cartesian to Polar Form

- A complex number $z = x + jy$ in cartesian form is converted to polar form via

$$r = \sqrt{x^2 + y^2}$$

and

$$\tan(\phi) = \frac{y}{x}.$$

- The computation of the angle ϕ requires some care.
- One must distinguish between the cases $x < 0$ and $x > 0$.

$$\phi = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \end{cases}$$

- If $x = 0$, ϕ equals $+\pi/2$ or $-\pi/2$ depending on the sign of y.

Exercise

- Convert to polar form
  1. $z = 1 + j$
  2. $z = 3 \cdot j$
  3. $z = -1 - j$
- Convert to cartesian form
  1. $z = 3e^{-j3\pi/4}$

Lecture: Complex Algebra

Introduction

- All *normal* rules of algebra apply!
- One difference to look for: $j \cdot j = -1$.
- Some operations are best carried out in rectangular coordinates.
  - Addition and subtraction
  - Multiplication and division aren't very hard, either.
- Others are easier in polar coordinates.
  - Multiplication and division.
  - Powers and roots
- New operation: conjugate complex.
- Slightly more subtle: absolute value.

Conjugate Complex

- The *conjugate complex* z^* of a complex number z has
  - the same real part as z: $\text{Re}\{z\} = \text{Re}\{z^*\}$, and
  - the opposite imaginary part: $\text{Im}\{z\} = -\text{Im}\{z^*\}$.
- **Rectangular form:**

$$\text{If } z = x + jy \text{ then } z^* = x - jy.$$

- **Polar form:**

$$\text{If } z = r \cdot e^{j\phi} \text{ then } z^* = r \cdot e^{-j\phi}.$$

- Note, z and z^* are mirror images of each other in the complex plane with respect to the real axis.

Addition and Subtraction

- Addition and subtraction can only be done in rectangular form.
  - If the complex numbers to be added are in polar form convert to rectangular form, first.
- Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$.
- **Addition:**

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- **Subtraction:**

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- Complex addition works like *vector addition*.

Multiplication

- Multiplication of complex numbers is possible in both polar and rectangular form.
- **Polar Form:** Let $z_1 = r_1 \cdot e^{j\phi_1}$ and $z_2 = r_2 \cdot e^{j\phi_2}$, then

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \exp(j(\phi_1 + \phi_2)).$$

- **Rectangular Form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + jy_1) \cdot (x_2 + jy_2) \\ &= x_1 x_2 + j^2 y_1 y_2 + jx_1 y_2 + jx_2 y_1 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

- Polar form provides more insight: multiplication involves rotation in the complex plane (because of $\phi_1 + \phi_2$).

Absolute Value

- The absolute value of a complex number z is defined as
$$|z| = \sqrt{z \cdot z^*}, \text{ thus, } |z|^2 = z \cdot z^*.$$
  - Note, $|z|$ and $|z|^2$ are real-valued.
  - In MATLAB, `abs(z)` computes $|z|$.
- **Polar Form:** Let $z = r \cdot e^{j\phi}$,
$$|z|^2 = r \cdot e^{j\phi} \cdot r \cdot e^{-j\phi} = r^2.$$
  - Hence, $|z| = r$.
- **Rectangular Form:** Let $z = x + jy$,
$$\begin{aligned} |z|^2 &= (x + jy) \cdot (x - jy) \\ &= x^2 - j^2 y^2 - jxy + jxy \\ &= x^2 + y^2. \end{aligned}$$

Division

- Closely related to multiplication

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}.$$

- **Polar Form:** Let $z_1 = r_1 \cdot e^{j\phi_1}$ and $z_2 = r_2 \cdot e^{j\phi_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \exp(j(\phi_1 - \phi_2)).$$

- **Rectangular Form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{z_1 z_2^*}{|z_2|^2} \\ &= \frac{(x_1+jy_1)\cdot(x_2-jy_2)}{x_2^2+y_2^2} \\ &= \frac{(x_1x_2+y_1y_2)+j(-x_1y_2+x_2y_1)}{x_2^2+y_2^2}. \end{aligned}$$

Exercises

- For $z_1 = 3e^{j\pi/4}$ and $z_2 = 2e^{-j\pi/2}$, compute
  1. $z_1 + z_2$,
  2. $z_1 \cdot z_2$, and
  3. $|z_1|$.

  Give your results in both polar and rectangular forms.

Lecture: Complex Algebra - Continued

Good to know ...

- You should try and remember the following relationships and properties.
    - $e^{j2\pi} = 1$
    - $e^{j\pi} = -1$
    - $e^{j\pi/2} = j$
    - $e^{-j\pi/2} = -j$
    - $|e^{j\phi}| = 1$ for all values of ϕ
    - $\exp(j(\phi + 2\pi)) = e^{j\phi}$

Powers of Complex Numbers

- A complex number z is easily raised to the n-th power if z is in polar form.
- Specifically,

$$\begin{aligned} z^n &= (r \cdot e^{j\phi})^n \\ &= r^n \cdot e^{jn\phi} \end{aligned}$$

  - The magnitude r is raised to the n-th power
  - The phase ϕ is multiplied by n.
- The above holds for arbitrary values of n, including
  - n an integer (e.g., z^2),
  - n a fraction (e.g., $z^{1/2} = \sqrt{z}$)
  - n a negative number (e.g., $z^{-1} = 1/z$)
  - n a complex number (e.g., z^j)

Roots of Unity

- Quite often all complex numbers z solving the following equation must be found

$$z^N = 1.$$

  - Here N is an integer.
  - There are N different complex numbers solving this equation.
- The solutions have the form

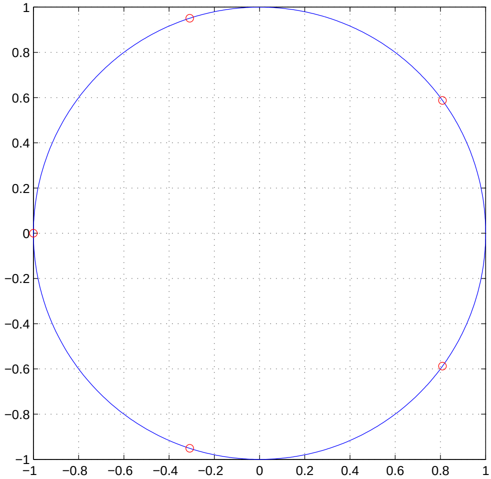
$$z = e^{j2\pi n/N} \text{ for } n = 0, 1, 2, \ldots, N-1.$$

  - Note that $z^N = e^{j2\pi n} = 1$!
- The solutions are called the N-th roots of unity.
- In the complex plane, all solutions have length 1 and are separated by angle $2\pi/N$.

Roots of a Complex Number

- The more general problem is to find *all* solutions of the equation
$$z^N = r \cdot e^{j\phi}.$$
- In this case, the N solutions are given by
$$z = r^{1/N} \cdot \exp(j\frac{\phi + 2\pi n}{N}) \text{ for } n = 0, 1, 2, \ldots, N-1.$$
- **Example:** Find all solutions of $z^5 = -1$.
- **Solution:**
  - Note $-1 = e^{j\pi}$, i.e., $r = 1$ and $\phi = \pi$.
  - The five solutions
    - all have magnitude 1,
    - and angles $\pi/5$, $3\pi/5$, $5\pi/5 = \pi$, $7\pi/5$, $9\pi/5$.

Roots of a Complex Number



Two Ways to Express $\cos(\phi)$

- First relationship: $\cos(\phi) = \text{Re}\{e^{j\phi}\}$
- Second relationship (inverse Euler):

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}.$$

- The first form is best suited as the starting point for problems involving the cosine or sine of a sum.
  - $\cos(\alpha + \beta)$
- The second form is best when products of sines and cosines are needed
  - $\cos(\alpha) \cdot \cos(\beta)$
- Rule of thumb: look to create products of exponentials.

Example

- Show that $\cos(x+y)$ equals $\cos(x)\cos(y) - \sin(x)\sin(y)$:

$$\begin{aligned}
\cos(x+y) &= \mathrm{Re}\{e^{j(x+y)}\} = \mathrm{Re}\{e^{jx} \cdot e^{jy}\} \\
&= \mathrm{Re}\{(\cos(x) + j\sin(x)) \cdot (\cos(y) + j\sin(y))\} \\
&= \mathrm{Re}\{(\cos(x)\cos(y) - \sin(x)\sin(y)) + \\
&\quad j(\cos(x)\sin(y) + \cos(y)\sin(y))\} \\
&= \cos(x)\cos(y) - \sin(x)\sin(y).
\end{aligned}$$

Example

- Show that $\cos(x)\cos(y)$ equals $\frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)$:

$$\begin{aligned}
\cos(x)\cos(y) &= \frac{e^{jx}+e^{-jx}}{2}\frac{e^{jy}+e^{-jy}}{2} \\
&= \frac{e^{j(x+y)}+e^{j(-x-y)}+e^{j(x-y)}+e^{j(-x+y)}}{4} \\
&= \frac{e^{j(x+y)}+e^{-j(x+y)}}{4}+\frac{e^{j(x-y)}+e^{-j(x-y)}}{4} \\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y).
\end{aligned}$$

Exercises

- Simplify
  1. $(\sqrt{2} - \sqrt{2}j)^8$
  2. $(\sqrt{2} - 2j)^{-1}$
- Advanced
  1. j^j
  2. $\cos(j)$

Lecture: Complex Exponentials

Introduction

- The complex exponential signal is defined as

$$x(t) = A\exp(j(2\pi ft + \phi)).$$

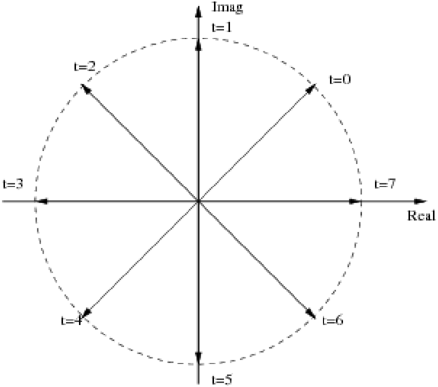
  - As with sinusoids, A, f, and ϕ are (real-valued amplitude, frequency, and phase.
- By Euler's relationship, it is closely related to sinusoidal signals

$$x(t) = A\cos(2\pi ft + \phi) + jA\sin(2\pi ft + \phi).$$

- We will apply the benefits the complex representation provides over sinusoids:
  - Avoid trigonometry,
  - Replace with simple algebra,
  - Visualization in the complex plane.

Complex Plane

$$x(t) = 1 \cdot \exp(j(\pi/4t + \pi/4))$$



Expressing Sinusoids through Complex Exponentials

- There are two ways to write a sinusoidal signal in terms of complex exponentials.
- **Real part:**

$$A\cos(2\pi ft+\phi)=\mathrm{Re}\{A\exp(j(2\pi ft+\phi))\}.$$

- **Inverse Euler:**

$$A\cos(2\pi ft+\phi)=\frac{A}{2}(\exp(j(2\pi ft+\phi))+\exp(-j(2\pi ft+\phi)))$$

- Both expressions are useful and will be important throughout the course.

Phasors

- Phasors are **not** directed-energy weapons first seen in the original Star Trek movie.
  - That would be *phas**e**rs*!
- Phasors are the complex amplitudes of complex exponential signals:

$$x(t) = A\exp(j(2\pi ft + \phi)) = Ae^{j\phi}\exp(j2\pi ft).$$

- The phasor of this complex exponential is $X = Ae^{j\phi}$.
- Thus, phasors capture both amplitude A and phase ϕ.

From Sinusoids to Phasors

- A sinusoid can be written as

$$A\cos(2\pi ft+\phi)=\frac{A}{2}(\exp(j(2\pi ft+\phi))+\exp(-j(2\pi ft+\phi))).$$

- This can be rewritten to provide

$$A\cos(2\pi ft+\phi)=\frac{Ae^{j\phi}}{2}\exp(j2\pi ft)+\frac{Ae^{-j\phi}}{2}\exp(-j2\pi ft).$$

- A sinusoid is composed of two complex exponentials
  - One with frequency f and phasor $\frac{Ae^{j\phi}}{2}$, and
  - One with frequency $-f$ and phasor $\frac{Ae^{-j\phi}}{2}$.
  - Note that the two phasors are conjugate complexes of each other.

Exercise

- Write
  $$x(t) = 3\cos(2\pi 10t - \pi/3)$$
  as a sum of two complex exponentials.
- For each of the two complex exponentials, find the frequency and the phasor.

What does this MATLAB code do?

```
NumPoints = 500;
tt = ( 0 : NumPoints ) / NumPoints; % tt goes from 0 to 1
UnitCircle = exp(j*2*pi*tt);

plot(UnitCircle)
axis('square')
```

MATLAB Scripts

- MATLAB scripts simply contain a sequence of MATLAB commands.
- They behave exactly as if the sequence of commands was typed in the command window.
  - All variables in the workspace can be accessed by the script.
  - New variables created by the script are visible in the workspace.
- Get in the habit of documenting your scripts:
  - At a minimum the first line should be of the form

```
% ScriptName Very brief decscription of script
```

  - This makes your script available to MATLAB's help system.
  - A more detailed description should follow immediately.

The Full MATLAB script

```
% PlotUnitCircle  Script file to plot a circle of radius one
%
% A circle of radius one is plotted in the current figure window. The
% script relies on the fact that exp(j*2*pi*t) defines a unit circle in the
% complex plane. Furthermore, it exploits that the plot command with a
% single complex-valued argument plots the real versus the imaginary part.
%
% Syntax:
%       PlotUnitCircle

NumPoints = 500;
tt = ( 0 : NumPoints ) / NumPoints; % tt goes from 0 to 1
UnitCircle = exp(j*2*pi*tt);

plot(UnitCircle)
axis('square')
```

MATLAB Functions

- MATLAB's functionality can be expanded by writing your own functions.
- Follow these rules to write a new function:
  - The very first line must be of the form

    ```
    function [out1 , out2] = MyFunction(in1 , in2)
    ```

  - The keyword `function` is required.
  - A vector of formal output parameters follows the word `function`.
    - No brackets are required for functions with only one output.
  - After the equal sign follows the name of the function.
    - The function must be stored in a file with the name of the function followed by '.m' (here `MyFunction.m`).
  - A list of formal input parameters follows in parentheses.
- Document your function!

The Header of a MATLAB function

```
function y = DoubleMe(x)
% DoubleMe - double the value of the input
%
% This function doubles the value of its input. The input
% may be a scalar, vector, or matrix and the result will
% be of the same dimension as the input.
%
% Syntax:
% y = DoubleMe(x)
%
```

The Body of a MATLAB Function

- Inside a MATLAB function, workspace variables are not available.
    - Any variables needed inside the function must be passed as input parameters.
- All variables inside a function are local; they disappear when the function finishes.
    - Variables needed in the workspace must be passed as output parameters.
- All output variables must be given a value.
- It is good coding practice to check that a function is given the correct input values.

The Body of a MATLAB function

```
% Check inputs
if nargin ~=1
    error('Function␣DoubleMe␣requires␣exactly␣one␣input.');
end
if nargout > 1
    error ('Function␣DoubleMe␣can␣have␣at␣most␣one␣output␣argument.');
end

% Compute result
y = 2*x;
```

Lecture: The Phasor Addition Rule

Problem Statment

- It is often required to add two or more sinusoidal signals.
- When all sinusoids have the same frequency then the problem simplifies.
  - This problem comes up in AC circuit analysis (ECE 280) and later in the class (chapter 5).
- Starting point: sum of sinusoids

$$x(t) = A_1 \cos(2\pi f t + \phi_1) + \ldots + A_N \cos(2\pi f t + \phi_N)$$

  - Note that all frequencies *f* are the same (no subscript).
  - Amplitudes A_i phases ϕ_i are different in general.
  - Short-hand notation using summation symbol ($\sum$):

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i)$$

The Phasor Addition Rule

- The phasor addition rule implies that there exist an amplitude A and a phase ϕ such that

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) = A \cos(2\pi f t + \phi)$$

- **Interpretation:** The sum of sinusoids of the same frequency but different amplitudes and phases is
  - a single sinusoid of the same frequency.
  - The phasor addition rule specifies how the amplitude A and the phase ϕ depends on the original amplitudes A_i and ϕ_i.
- **Example:** We showed earlier (by means of an unpleasant computation involving trig identities) that:

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2) = 5 \cos(2\pi f t + 53^o)$$

Prerequisites

- We will need two simple prerequisites before we can derive the phasor addition rule.
  1. Any sinusoid can be written in terms of complex exponentials as follows
     $$A\cos(2\pi ft+\phi) = \text{Re}\{Ae^{j(2\pi ft+\phi)}\} = \text{Re}\{Ae^{j\phi}e^{j2\pi ft}\}.$$
     Recall that $Ae^{j\phi}$ is called a phasor (complex amplitude).
  2. For any complex numbers $X_1, X_2, \ldots, X_N$, the real part of the sum equals the sum of the real parts.
     $$\text{Re}\left\{\sum_{i=1}^{N} X_i\right\} = \sum_{i=1}^{N}\text{Re}\{X_i\}.$$
     - This should be obvious from the way addition is defined for complex numbers.
       $$(x_1+jy_1)+(x_2+jy_2) = (x_1+x_2)+j(y_1+y_2).$$

Deriving the Phasor Addition Rule

- **Objective**: We seek to establish that

$$\sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) = A \cos(2\pi f t + \phi)$$

  and determine how A and ϕ are computed from the A_i and ϕ_i.

- **Step 1:** Using the first pre-requisite, we replace the sinusoids with complex exponentials

$$\begin{aligned} \textstyle\sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i) &= \textstyle\sum_{i=1}^{N} \mathrm{Re}\{A_i e^{j(2\pi f t + \phi_i)}\} \\ &= \textstyle\sum_{i=1}^{N} \mathrm{Re}\{A_i e^{j\phi_i} e^{j2\pi f t}\}. \end{aligned}$$

Deriving the Phasor Addition Rule

- **Step 2:** The second prerequisite states that the sum of the real parts equals the the real part of the sum

$$\sum_{i=1}^{N} \mathrm{Re}\{A_i e^{j\phi_i} e^{j2\pi ft}\} = \mathrm{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\}.$$

- **Step 3:** The exponential $e^{j2\pi ft}$ appears in all the terms of the sum and can be factored out

$$\mathrm{Re}\left\{\sum_{i=1}^{N} A_i e^{j\phi_i} e^{j2\pi ft}\right\} = \mathrm{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\}$$

Deriving the Phasor Addition Rule

- The term $\sum_{i=1}^{N} A_i e^{j\phi_i}$ is just the sum of complex numbers in polar form.
- The sum of complex numbers is just a complex number X which can be expressed in polar form as $X = Ae^{j\phi}$.
- Hence, amplitude A and phase ϕ must satisfy

$$Ae^{j\phi} = \sum_{i=1}^{N} A_i e^{j\phi_i}$$

- **Step 4:** Substituting into our last expression yields:

$$\begin{aligned}\mathrm{Re}\left\{\left(\sum_{i=1}^{N} A_i e^{j\phi_i}\right) e^{j2\pi ft}\right\} &= \mathrm{Re}\left\{Ae^{j\phi} e^{j2\pi ft}\right\} \\ &= \mathrm{Re}\left\{Ae^{j(2\pi ft+\phi)}\right\} \\ &= A\cos(2\pi ft + \phi).\end{aligned}$$

Applying the Phasor Addition Rule

- Applicable only when sinusoids of same frequency need to be added!
- **Problem:** Simplify

$$x(t) = A_1 \cos(2\pi f t + \phi_1) + \ldots A_N \cos(2\pi f t + \phi_N)$$

- **Solution:** proceeds in 4 steps
  1. Extract phasors: $X_i = A_i e^{j\phi}$ for $i = 1, \ldots, N$.
  2. Convert phasors to rectangular form: $X_i = A_i \cos\phi_i + jA_i \sin\phi_i$ for $i = 1, \ldots, N$.
  3. Compute the sum: $X = \sum_{i=1}^{N} X_i$ by adding real parts and imaginary parts, respectively.
  4. Convert result X to polar form: $X = Ae^{j\phi}$.
- **Conclusion:** With amplitude A and phase ϕ determined in the final step

$$x(t) = A\cos(2\pi f t + \phi).$$

Example

- **Problem:** Simplify

$$x(t) = 3 \cdot \cos(2\pi f t) + 4 \cdot \cos(2\pi f t + \pi/2)$$

- **Solution:**
  1. Extract Phasors: $X_1 = 3e^{j0} = 3$ and $X_2 = 4e^{j\pi/2}$.
  2. Convert to rectangular form: $X_1 = 3$ $X_2 = 4j$.
  3. Sum: $X = X_1 + X_2 = 3 + 4j$.
  4. Convert to polar form: $A = \sqrt{3^2 + 4^2} = 5$ and $\phi = \arctan(\frac{4}{3}) \approx 53^o$ $(\frac{53}{180}\pi)$.

- **Result:**

$$x(t) = 5\cos(2\pi f t + 53^o).$$

Exercise

- Simplify

$$x(t) = 10\cos(20\pi t + \frac{\pi}{4}) + 10\cos(20\pi t + \frac{3\pi}{4}) + 20\cos(20\pi t - \frac{3\pi}{4}).$$

- Answer:

$$x(t) = 10\sqrt{2}\cos(20\pi t + \pi).$$