Filtering in the Frequency Domain 12/6/16

Reminder:

$$x[n] = e^{j2\pi \hat{f} n} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n] = H(\hat{f}) \cdot e^{j2\pi \hat{f} n}$$

where: $$\boxed{H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi \hat{f} k}}$$

Question: How can we leverage the above insight for filtering of arbitrary signals?

Starting Point: Spectrum Representation

Any signal $x[n]$ can always be written as a sum of complex exponentials:

$$x[n] = \sum_{k=-\infty}^{\infty} X_k \cdot e^{j2\pi f_k n}$$

$\Rightarrow$ Spectrum of signal $x[n]$:

$$X(f) = \{(X_0, f_0), (X_1, f_1), \ldots\}$$

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Ultimately, want to $\overset{\text{filter}}{\vee}$ $x[n]$.

Begin by passing individual complex exponentials through filter

$$\mathcal{X}_0 \cdot e^{j2\pi f_0 n} \longmapsto \mathcal{X}_0 \cdot H(f_0) \cdot e^{j2\pi f_0 n}$$

$$\mathcal{X}_1 \cdot e^{j2\pi f_1 n} \longmapsto \mathcal{X}_1 \, H(f_1) \, e^{j2\pi f_1 n}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$+ \quad \overline{} \qquad \overline{}$$

$$x[n] = \sum_k \mathcal{X}_k \, e^{j2\pi f_k \cdot n} \longmapsto \sum_k \mathcal{X}_k \cdot H(f_k) \cdot e^{j2\pi f_k \cdot n} = y[n]$$

<u>Note:</u> Spectrum of the output signal

$$Y(f) = \{ (\mathcal{X}_0 \cdot H(f_0), f_0), (\mathcal{X}_1 H(f_1), f_1), \ldots \}$$

This is written concisely as

$$\boxed{Y(f) = \mathcal{X}(f) \cdot H(f)}$$

Example:

$$x[n] = 2\cos(2\pi 0.25 n)\cdot\cos(2\pi 0.1 n)$$

$$h[n] = \{\underset{\uparrow}{1}, 1, 1\}$$

Find $y[n]$

Plan:

1. ~~Find spectrum of $x[n]$~~
2. ~~Find frequency response~~
3. ~~Find output spectrum $Y(f)$~~
4. Find $y[n]$

1 Find $X(f)$:

$$x[n] = 2\cos(2\pi 0.25 n).\cos(2\pi 0.1 n)$$

$$= \cancel{2}\cdot\left(\frac{e^{j2\pi 0.25 n} + e^{-j2\pi 0.25 n}}{\cancel{2}}\right)\cdot\left(\frac{e^{j2\pi 0.1 n} + e^{-j2\pi 0.1 n}}{2}\right)$$

$$= \frac{1}{2}\cdot\left[e^{j2\pi 0.35 n} + e^{-j2\pi 0.35 n} + e^{j2\pi 0.15 n} + e^{-j2\pi 0.15}\right.$$

$$\Rightarrow \boxed{X(f) = \left\{\left(\tfrac{1}{2}, 0.35\right), \left(\tfrac{1}{2}, -0.35\right), \left(\tfrac{1}{2}, 0.15\right), \left(\tfrac{1}{2}, -0.15\right)\right\}}$$

(2) Find $H(\hat{f})$: $h[n] = \{1, 1, 1\}$

$$H(\hat{f}) = \sum_{k=0}^{2} h[k] \cdot e^{-j2\pi \hat{f} k}$$

$$= 1 + 1 \cdot e^{-j2\pi\hat{f}} + 1 \cdot e^{-j2\pi\hat{f} \cdot 2}$$

$$= e^{-j2\pi\hat{f}} \cdot \left(e^{j2\pi\hat{f}} + 1 + e^{-j2\pi\hat{f}} \right)$$

$$\boxed{= e^{-j2\pi\hat{f}} \cdot \left(1 + 2\cos(2\pi\hat{f}) \right)}$$

(3) $Y(f) = H(f) \cdot X(f)$

Evaluate $H(f)$ at frequencies that occur in $X(f)$.

$\hat{f} = 0.35$: $H(0.35) = e^{-j0.7\pi} \cdot (1 + 2\cos(0.7\pi))$

$$= e^{-j0.7\pi} \cdot (-0.18) = 0.18 \cdot e^{j0.3\pi}$$

$\hat{f} = -0.35$: $H(-0.35) = e^{j0.7\pi} \cdot (1 + 2\cos(-0.7\pi))$

$$= e^{j0.7\pi} \cdot (-0.18) = 0.18 \cdot e^{-j0.3\pi}$$

$\hat{f} = 0.15$: $H(0.15) = e^{-j0.3\pi} \cdot (1 + 2\cos(0.3\pi))$

$$= 2.18 \cdot e^{-j0.3\pi}$$

$\hat{f} = -0.15$: $H(-0.15) = 2.18 \cdot e^{j0.3\pi}$

Multiply $X(f_k)$ and $H(f_k)$ to obtain $Y(f_k)$:

$$\hat{f} = 0.35: \; Y(\hat{f}) = \frac{1}{2} \cdot 0.18 \cdot e^{j0.3\pi}$$
$$= 0.09 \cdot e^{j0.3\pi}$$

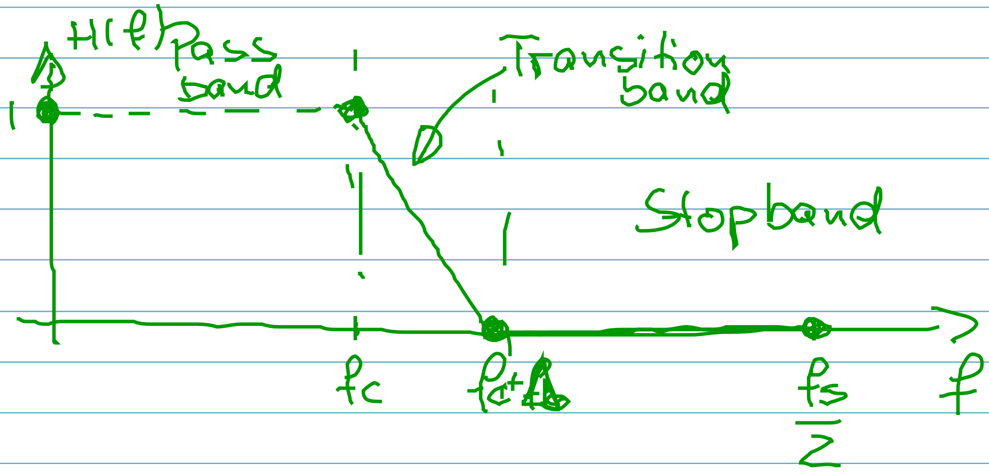
$$\hat{f} = -0.35: \; Y(\hat{f}) = 0.09 \cdot e^{-j0.3\pi}$$

$$\hat{f} = 0.15: \; Y(\hat{f}) = \frac{1}{2} \cdot 2.18 \; e^{-j0.3\pi} = 1.09 \cdot e^{-j0.3\pi}$$
$$\hat{f} = -0.15: \; Y(\hat{f}) = 1.09 \cdot e^{j0.3\pi}$$

$$Y(\hat{f}) = \{ (\underline{0.09} \cdot e^{\underline{j0.3\pi}}, \underline{0.35}), (0.09\, e^{-j0.3\pi}, -0.35), (1.09\, e^{-j0.3\pi}, 0.15), (1.09 \cdot e^{+j0.3\pi}, -0.15) \}$$

(4)
$$y[n] = \underline{0.09} \cdot e^{j0.3\pi} \cdot e^{j2\pi \underline{0.35} n} + 0.09 \cdot e^{-j0.3\pi} \, e^{-j2\pi 0.35 n} + 1.09 \cdot e^{-j0.3\pi} \cdot e^{j2\pi 0.15 n} + 1.09 \cdot e^{+j0.3\pi} \cdot e^{-j2\pi 0.15 n}$$
$$= 0.18 \cos(2\pi 0.35 n + 0.3\pi) + 2.18 \cos(2\pi 0.15 n - 0.3\pi)$$

Designing filters in MATLAB:



freq: [0 , fc , fc+Δ , fs/2]

mag: [1 , 1 , 0 , 0]

hh = firpm(100, freq/(fs/2), mag)

freqz(hh, fs) → plots frequency response

$$\text{Mag,dB} = 10 \cdot \log_{10}(|H(f)|^2)$$