12/1/16 Frequency Response of LTI System

For an LTI system with impulse response $h[n]$, find the frequency response $H(\hat{f})$.

$$e^{j2\pi\hat{f}n} \longrightarrow \boxed{h[n]} \longrightarrow ?\ y[n] = ?$$

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$x[n] = e^{j2\pi\hat{f}n}$:

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j2\pi\hat{f}(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j2\pi\hat{f}n} \cdot e^{-j2\pi\hat{f}k}$$

$$= \boxed{e^{j2\pi\hat{f}n}} \cdot \boxed{\sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi\hat{f}k}}$$

Output signal $y[n]$ is a complex exponential of same frequency $\hat{f}$ as $x[n]$

Frequency response $H(\hat{f})$
- constant, for given $\hat{f}$
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12/11.6 Frequency Response of LTI Systems

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$$= \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j2\pi\hat{f}n} \, e^{-j2\pi\hat{f}k}$$

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Output signal $y[n]$ is a complex exponential of same frequency $\hat{f}$ as $x[n]$

Frequency response $H(\hat{f})$
- constant for given $\hat{f}$
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Relationship between impulse response $h[n]$ and frequency response $H(\hat{f})$:

$$\boxed{H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j2\pi\hat{f}k}}$$

Observations:

- For an LTI system, if $x[n]$ is a complex exponential of frequency $\hat{f}$ then output $y[n]$ is also a complex exponential of frequency $\hat{f}$.
- If $x[n] = A \cdot e^{j(2\pi\hat{f}n+\phi)}$ then

$$y[n] = H(\hat{f}) \cdot x[n]$$

- $$y[n] = \underbrace{A \cdot e^{j\phi} \cdot e^{j2\pi\hat{f}n}}_{=x[n]} \cdot H(\hat{f})$$

write $H(\hat{f})$ in polar form:

$$H(\hat{f}) = |H(\hat{f})| \cdot e^{j\angle H(\hat{f})}$$

Then,

$$y[n] = A \cdot e^{j\phi} \cdot e^{j2\pi\hat{f}n} \cdot |H(\hat{f})| \cdot e^{j\angle H(\hat{f})}$$

$$= \underbrace{A \cdot |H(\hat{f})|}_{\text{ampl of } y[n]} \cdot e^{j\overbrace{(\phi + \angle H(\hat{f}))}^{\text{phase of } y[n]}} \cdot e^{j2\pi\hat{f}n}$$

Examples:

First differencefilter

$$y[n] = x[n] - x[n-1]$$

$$\Rightarrow h[n] = \{1, -1\}$$

$$\Rightarrow H(\hat{f}) = \sum_{k=0}^{1} h[k] \cdot e^{-j2\pi \hat{f} k}$$

$$= 1 \cdot e^{j \cdot 0} - 1 \cdot e^{-j2\pi \hat{f} \cdot 1}$$

$$= 1 - e^{-j2\pi \hat{f}}$$

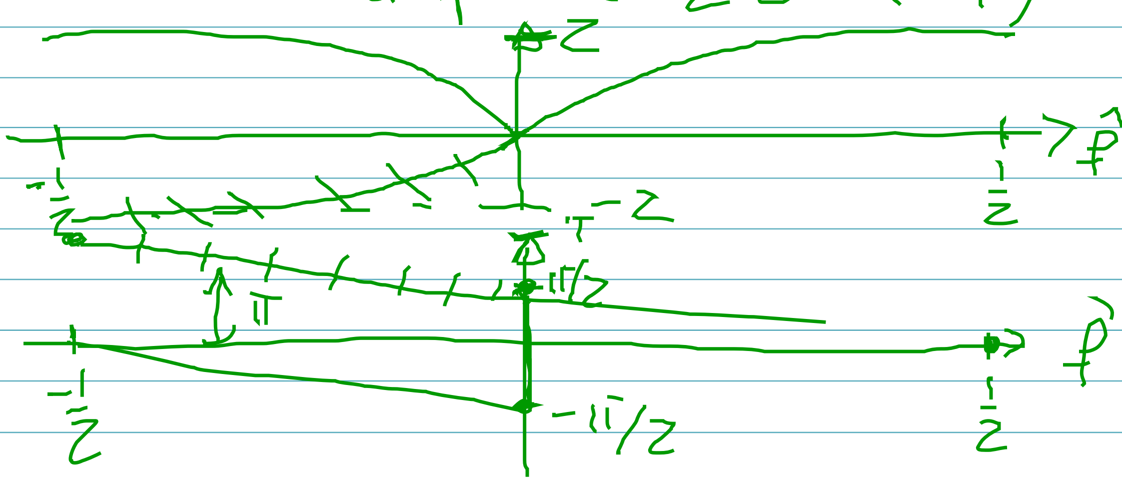
$$= e^{-j\pi \hat{f}} \cdot \underbrace{\left(e^{j\pi \hat{f}} - e^{-j\pi f}\right)}_{\substack{= 2j \sin(\pi f) \\ = 2e^{j\pi/2} \cdot \sin(\pi f)}}$$

$$e^{-j(\pi \hat{f} - \pi/2)}$$

$$= e^{-j(\hat{\pi f} - \pi/2)} \cdot 2 \sin(\pi f)$$

phase: $-(\pi \hat{f} - \frac{\pi}{2})$

amplitude: $2 \sin(\pi f)$



$$y[n] = x[n] - 2x[n-1] + x[n-2]$$

$$h[n] = \{1, -2, +1\}$$

$$H(\hat{f}) = \sum_{k=0}^{2} h[k] \cdot e^{-j2\pi\hat{f}k}$$

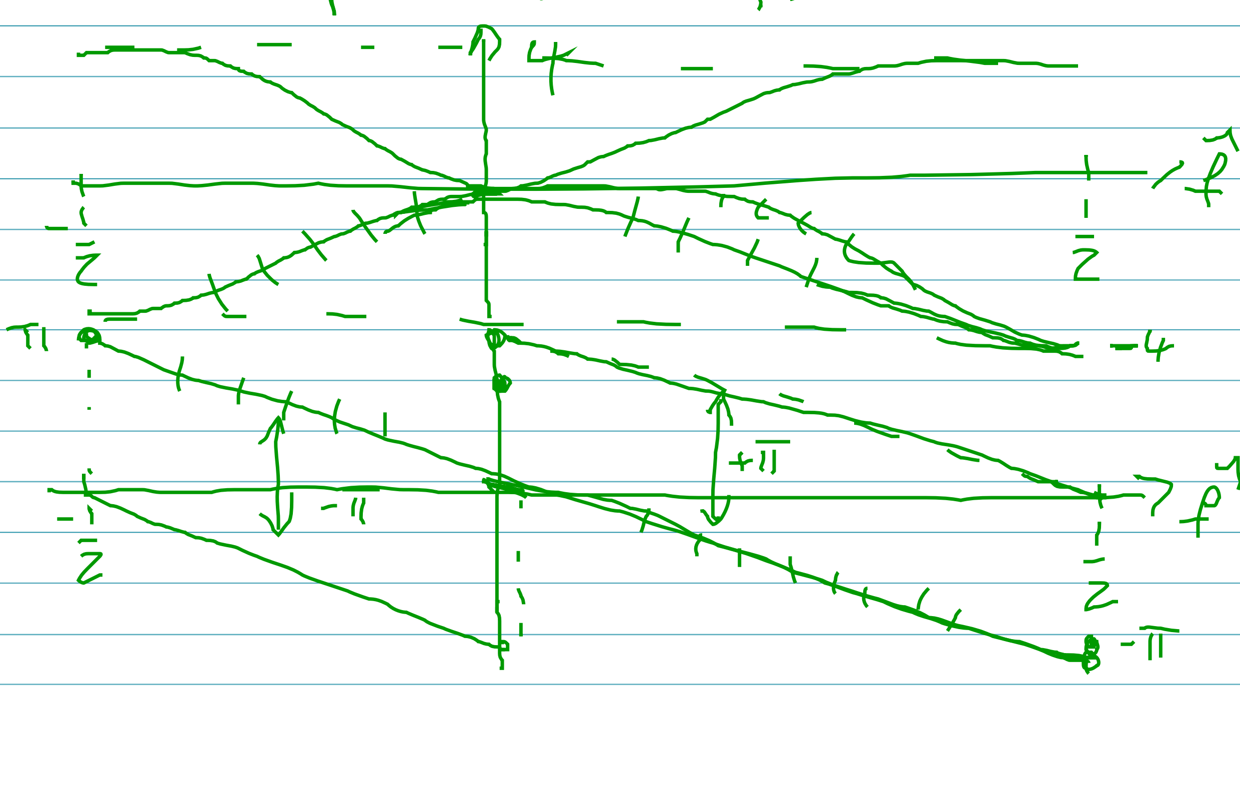
$$= \underset{h[0]}{1} - \underset{h[1]}{2} \cdot e^{-j2\pi\hat{f}} + \underset{h[2]}{1} \cdot e^{-j4\pi\hat{f}}$$

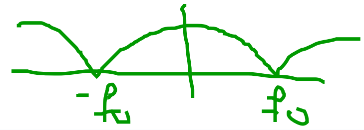
$$= e^{-j2\pi\hat{f}} \cdot \Big(\underbrace{e^{j2\pi\hat{f}} - 2 + e^{-j2\pi\hat{f}}}_{=2\cos(2\pi\hat{f})} \Big)$$

$$= e^{-j2\pi\hat{f}} \cdot \left(2\cos(2\pi\hat{f}) - 2\right)$$

Phase: $-2\pi\hat{f}$

Amplitude: $2\cos(2\pi\hat{f}) - 2$





<u>Question:</u> Can we make a filter that rejects a given frequency $\pm f_0$

$$\Rightarrow H(\hat{f}) = \left(e^{-j2\pi f_0} - e^{-j2\pi \hat{f}}\right)\cdot\left(e^{j2\pi f_0} - e^{-j2\pi \hat{f}}\right)$$

$$= 1 - e^{-j2\pi(\hat{f}+f_0)} - e^{-j2\pi(\hat{f}-f_0)} + e^{-j4\pi \hat{f}}$$

$$= 1 - e^{-j2\pi \hat{f}}\cdot\underbrace{\left(e^{-j2\pi f_0} + e^{+j2\pi f_0}\right)}_{=2\cos(2\pi f_0)} + e^{-j4\pi \hat{f}}$$

$$= 1 - 2\cos(2\pi f_0)\, e^{-j2\pi \hat{f}} + e^{-j4\pi \hat{f}}$$

Compare to Frequency response of $h[n] = \{b_0, b_1, \}$

$$\Rightarrow b_0 \cdot 1 + b_1 \cdot e^{-j2\pi \hat{f}} + b_2 \cdot e^{-j4\pi \hat{f}}$$

<u>Conclusion:</u> Filter must have impulse response:

$$h[n] = \{1, -2\cos(2\pi f_0), 1\}$$