HW 7 - Solution

The following prerequisites are helpful for the remaining problems:

Prerequisites:

$$R_t = s(t) + N_t$$

$s(t)$ - known

N_t - WGN spectral height $\frac{N_0}{2}$

RX frontend:

$$R = \langle R_t, g(t) \rangle = \int R_t \cdot g(t)\, dt$$

$\Rightarrow R$ is Gaussian (linearity of $\langle \cdot, \cdot \rangle$)

mean: $\mu = E[R \mid s(t)]$

$$= E[\langle R_t, g(t)\rangle \mid s(t)]$$

$$= E[\langle s(t) + N_t, g(t)\rangle]$$

$$= \langle s(t) + \underbrace{E[N_t]}_{=0}, g(t)\rangle] = \underline{\langle s(t), g(t)\rangle}$$

Variance $\sigma^2 = E[(R - E[R \mid s(t)])^2 \mid s(t)]$

$$= E[(\langle R_t, g(t)\rangle - \langle s(t), g(t)\rangle)^2 \mid s(t)]$$

$$= E[(\langle \cancel{s(t)} + N_t, g(t)\rangle - \cancel{\langle s(t), g(t)\rangle})^2]$$

$$= E[(\langle N_t, g(t)\rangle)^2]$$

$$= E[\langle N_t, g(t)\rangle \cdot \langle N_u, g(u)\rangle]$$

$$= E\left[\int N_t\, g(t)\, dt \cdot \int N_u\, g(u)\, du\right]$$

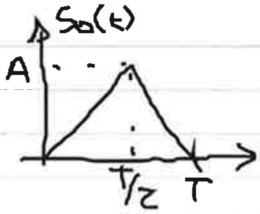
$E[N_t \cdot N_u] = \frac{N_0}{2}\delta(t-u)$

$$= \iint E[N_t \cdot N_u]\, g(t)\, g(u)\, dt\, du$$

$$= \frac{N_0}{2} \int \underbrace{\int \delta(t-u)\, g(t)\, dt}_{=g(u)}\, g(u)\, du$$

$$= \frac{N_0}{2} \int g^2(u)\, du = \frac{N_0}{2} \cdot \|g(t)\|^2$$

1. $s_0(t) = \begin{cases} 2At/T & 0 \le t \le T/2 \\ 2A - 2At/T & T/2 \le t \le T \\ 0 & \text{else} \end{cases}$



$s_1(t) = -s_0(t)$ priors $\pi_0 = \pi_1 = 1/2$

$\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0 \end{cases}$ $A = \sqrt{\frac{3E}{T}}$

a) Energy:

$$E_0 = \int_0^T s_0^2(t)\,dt = 2 \cdot \int_0^{T/2} s_0^2(t)\,dt$$

$$= 2 \cdot \int_0^{T/2} \left(\frac{2A}{T}\right)^2 \cdot t^2\,dt$$

$$= 2 \cdot \left(\frac{2A}{T}\right)^2 \cdot \frac{t^3}{3}\Bigg|_0^{T/2}$$

$$= \frac{8 \cdot A^2}{T^2} \cdot \frac{T^3/8}{3} = \frac{A^2 \cdot T}{3} = E$$

$E_1 = E_0$

b) $g(t) = 1$ for $0 \le t \le T$

$s_0(t)$: $E[R|s_0] = \frac{A \cdot T}{2} = \sqrt{\frac{3ET}{4}}$

$\mathrm{Var}[R|s_0] = \frac{N_0}{2} \cdot T$

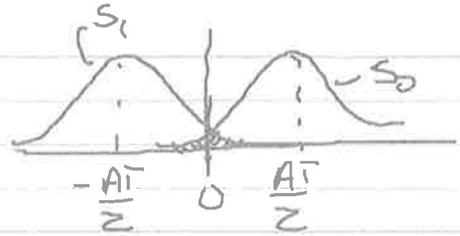
$s_1(t)$: $E[R|s_1] = \langle s_1(t), g(t) \rangle = -\frac{AT}{2} = -\sqrt{\frac{3ET}{4}}$

$\mathrm{Var}[R|s_1] = \frac{N_0}{2} T$

$$P_e = \pi_0 \cdot P_{e|s_0} + \pi_1 \cdot P_{e|s_1}$$

$$= P_{e|s_1}$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{1}{2}\frac{(x+\mu)^2}{\sigma^2}\right) dx$$



$\mu = \frac{AT}{2}$

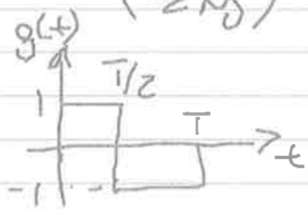
$\sigma^2 = \frac{N_0}{2} \cdot T$

$z = \frac{x+\mu}{\sigma}$ $\quad = \int_{\mu/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = Q\left(\frac{\mu}{\sigma}\right)$

$$= Q\left(\frac{AT}{2\sqrt{\frac{N_0 T}{2}}}\right) = Q\left(\frac{\sqrt{\frac{3ET}{4}}}{\sqrt{\frac{N_0}{2}T}}\right) =$$

$$= Q\left(\sqrt{\frac{3}{2}\frac{E}{N_0}}\right)$$

c) $g(t) = \begin{cases} 1 & 0 \le t \le T/2 \\ -1 & T/2 \le t \le T \\ 0 & \text{else} \end{cases}$



$E[R|s_0] = 0 \qquad E[R|s_1] = 0$

$\Rightarrow P_e = \frac{1}{2}$

d) $g(t) = s_0(t)$

$E[R|s_0] = \langle s_0(t), s_0(t) \rangle = \|s_0(t)\|^2 = E$

$Var[R|s_0] = \frac{N_0}{2} \cdot \|s_0(t)\|^2 = \frac{N_0}{2} \cdot E$

$E[R|s_1] = -E \qquad Var[R|s_1] = \frac{N_0}{2} \cdot E$

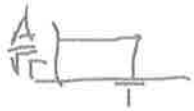
$\mu = E \qquad \sigma^2 = \frac{N_0}{2} \cdot E$

$$\Rightarrow P_e = Q\left(\frac{\mu}{\sigma}\right) = Q\left(\frac{E}{\sqrt{\frac{N_0}{2}E}}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

e) Receiver in (d) is best

Projects into space spanned by

$s_0(t)$ & $s_1(t)$.



2) $s_0(t) = A/\sqrt{T}$ $\quad 0 \le t \le T$ $\qquad s_1(t) = -1/\sqrt{T}$ $\quad 0 \le t \le$

$R = \langle R_t, g(t) \rangle$ with $g(t) = 1$ $\quad 0 \le t \le T$

$= \int R_t \, dt$ $\qquad \pi_0 = \pi_1 = \frac{1}{2}$

$$\hat{m} = \begin{cases} 0 & \text{if } R > \gamma \\ 1 & \text{if } R < \gamma \end{cases}$$

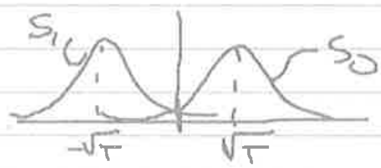
a) $A = 0$, $\gamma = 0$

$E[R|s_0] = \langle s_0(t), g(t) \rangle = A\sqrt{T} = 1\sqrt{T}$

$Var[R|s_0] = \frac{N_0}{2} \|g(t)\|^2 = \frac{N_0}{2} T$

$E[R|s_1] = -\sqrt{T}$ $\qquad Var[R|s_1] = \frac{N_0}{2} T$

$P_e = P_{e|s_1} = Q\left(\frac{\sqrt{T}}{\sqrt{\frac{N_0}{2} T}}\right) = Q\left(\sqrt{\frac{2}{N_0}}\right)$

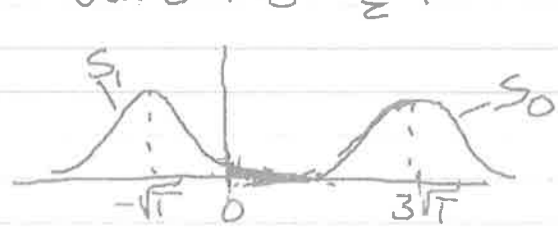


b) $A = 3$, $\gamma = 0$

$E[R|s_0] = 3\sqrt{T}$ $\qquad Var[R|s_0] = \frac{N_0}{2} T$

$E[R|s_1] = -\sqrt{T}$ $\qquad Var[R|s_1] = \frac{N_0}{2} T$

$P_e = \frac{1}{2} P_{e|s_1} + \frac{1}{2} P_{e|s_0}$



$P_{e|s_1} = Q\left(\sqrt{\frac{2}{N_0}}\right)$

$P_{e|s_0} = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2} \frac{(x - \mu_0)^2}{\sigma^2}\right) dx$ $\qquad \mu_0 = 3\sqrt{T}$, $\sigma^2 = \frac{N_0}{2} T$

$z=\frac{x-\mu_0}{\sigma}$: $\int_{-\infty}^{-\frac{\mu_0}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = \int_{\frac{\mu_0}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$

$$= Q\left(\frac{\mu_0}{\sigma}\right) = Q\left(\frac{3\sqrt{T}}{\sqrt{\frac{N_0}{2}\cdot T}}\right)$$

$$= Q\left(\sqrt{\frac{18}{N_0}}\right)$$

$$P_e = \frac{1}{2}\cdot\left(Q\left(\sqrt{\frac{2}{N_0}}\right) + Q\left(\sqrt{\frac{18}{N_0}}\right)\right)$$

c) $A=3$, better γ?

$$\gamma = \frac{1}{2}\left(3\sqrt{T} + (-\sqrt{T})\right) = \sqrt{T}$$

$$P_e = P_{e|s_1} = \int_{\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x+\sqrt{T})^2}{\sigma^2}\right) dx$$

$z=\frac{x+\sqrt{T}}{\sigma}$: $\int_{\frac{2\sqrt{T}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = Q\left(\frac{2\sqrt{T}}{\sigma}\right)$

$$= Q\left(\frac{2\sqrt{T}}{\sqrt{N_0/2\cdot T}}\right) = Q\left(\sqrt{\frac{8}{N_0}}\right)$$

d) $\gamma = \frac{1}{2}\left(A\cdot\sqrt{T} - \sqrt{T}\right) = \frac{A-1}{2}\cdot\sqrt{T}$

Then: $P_e = P_{e|s_0} = P_{e|s_1} = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x+\sqrt{T})^2}{\sigma^2}\right) dx$

$$= Q\left(\frac{\gamma+\sqrt{T}}{\sigma}\right)$$

$$= Q\left(\frac{A+1}{\sqrt{2N_0}}\right)$$

(3) $s_0(t) = \sqrt{\frac{E_b}{T}} \quad 0 \le t \le T \qquad s_1(t) = -s_0(t)$

$\pi_0 = \frac{3}{4} \qquad \pi_1 = \frac{1}{4}$

$R = \int R_t \, dt$

$\hat{m} = \begin{cases} 0 & R > \gamma \\ 1 & R < \gamma \end{cases}$

a) $\gamma = 0$

$E[R|s_0] = \sqrt{E_b \cdot T} \qquad E[R|s_1] = -\sqrt{E_b \cdot T}$

$Var[R|s] = \frac{N_0}{2} \cdot T$

$P_e = P_{e|s_1} = Q\left(\frac{\sqrt{E_b \cdot T}}{\sqrt{\frac{N_0}{2} \cdot T}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

b) γ

$P_e = \pi_0 \cdot P_{e|s_0} + \pi_1 \cdot P_{e|s_1} \leftarrow$

$P_{e|s_1} = \int_{\gamma}^{\infty} P_{R|s_1}(r)\, dr = Q\left(\frac{\gamma + \sqrt{E_b T}}{\sqrt{\frac{N_0}{2} \cdot T}}\right)$

$P_{e|s_1} = \int_{-\infty}^{\gamma} P_{R|s_0}(r)\, dr = Q\left(\frac{\sqrt{E_b \cdot T} - \gamma}{\sqrt{\frac{N_0}{2} \cdot T}}\right)$

$P(e) = \frac{3}{4} \cdot Q\left(\frac{\sqrt{E_b T} - \gamma}{\sqrt{N_0/2 \cdot T}}\right) + \frac{1}{4} Q\left(\frac{\sqrt{E_b T} + \gamma}{\sqrt{N_0/2 \cdot T}}\right)$

c) $\dfrac{dP_e(\gamma)}{d\gamma} \overset{!}{=} 0$ $\qquad \mu = \sqrt{E_b T}$

$\qquad \sigma^2 = \frac{N_0}{2} \cdot T$

$$P_e(\gamma) = \pi_0 \cdot Q\left(\frac{\mu-\gamma}{\sigma}\right) + \pi_1 \cdot Q\left(\frac{\mu+\gamma}{\sigma}\right)$$

$$\frac{dP_e(\gamma)}{d\gamma} = \pi_0 \cdot \left(-\frac{1}{\sigma}\right) \cdot \frac{dQ(x)}{dx}\Big|_{x=\frac{\mu-\gamma}{\sigma}} + \pi_1 \cdot \left(\frac{1}{\sigma}\right) \cdot \frac{dQ(x)}{dx}\Big|_{x=\frac{\mu+\gamma}{\sigma}}$$

$$Q(x) = \int_x^{\infty} f_N(z)\,dz = F_N(\infty) - F_N(x)$$

$$\Rightarrow \frac{dQ(x)}{dx} = 0 - f_N(x) = -f_N(x) \qquad f_N \text{ pdf for } N(0,1)$$

$$\frac{dP_e(\gamma)}{d\gamma} = -\frac{\pi_0}{\sigma} \cdot \left(-f_N\left(\frac{\mu-\gamma}{\sigma}\right)\right) + \frac{\pi_1}{\sigma} \cdot \left(-f_N\left(\frac{\mu+\gamma}{\sigma}\right)\right) \overset{!}{=} 0$$

$$\Rightarrow \frac{\pi_0}{\cancel{\sigma}} \cdot \frac{1}{\cancel{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}\frac{(\mu-\gamma)^2}{\sigma^2}\right) = \frac{\pi_1}{\cancel{\sigma}} \cdot \frac{1}{\cancel{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}\frac{(\mu+\gamma)^2}{\sigma^2}\right)$$

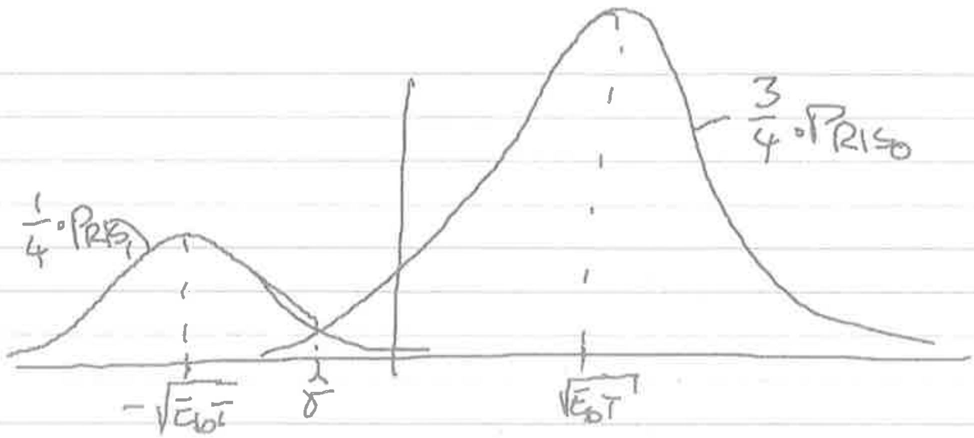
$$\ln: \Rightarrow \ln(\pi_0) - \frac{1}{2}\frac{(\mu-\gamma)^2}{\sigma^2} = \ln(\pi_1) - \frac{1}{2}\frac{(\mu+\gamma)^2}{\sigma^2}$$

$$\Rightarrow \left(\ln(\pi_0) - \ln(\pi_1)\right) \cdot 2\sigma^2 = \underbrace{(\mu-\gamma)^2}_{\mu^2 - 2\mu\gamma + \gamma^2} - \underbrace{(\mu+\gamma)^2}_{\mu^2 + 2\mu\gamma + \gamma^2}$$

$$\Rightarrow \ln\frac{\pi_0}{\pi_1} \cdot 2\sigma^2 = -4\mu \cdot \gamma$$

$$\Rightarrow \gamma = -\frac{\sigma^2}{2\mu} \cdot \ln\frac{\pi_0}{\pi_1} = \frac{\sigma^2}{2\mu} \cdot \ln\frac{\pi_1}{\pi_0}$$

$$= \frac{\frac{N_0}{2} \cdot T}{2\sqrt{E_b T}} \ln\frac{\pi_1}{\pi_0} = \frac{N_0}{4} \cdot \sqrt{\frac{T}{E_b}} \cdot \ln\frac{\pi_1}{\pi_0}$$



1/4 · PR|S1
3/4 · PR|S0
-√E0T
γ
√E0T

(4) $S_0: \quad R_t = W_t + N_t \qquad W_t \sim WGN\left(\frac{E}{2}\right)$

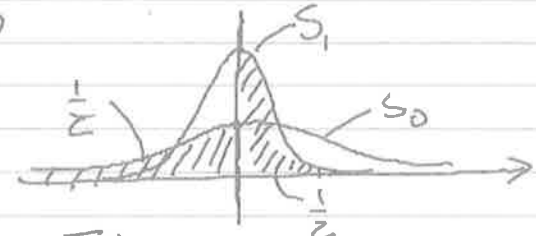
$S_1: \quad R_t = N_t \qquad N_t \sim WGN\left(\frac{N_0}{2}\right)$

$R = \int R_t \, dt \qquad \pi_0 = \pi_1 = \frac{1}{2}$

a) $E[R|S_0] = 0 \qquad Var[R|S_0] = \left(\frac{N_0}{2} + \frac{E}{2}\right) \cdot T$

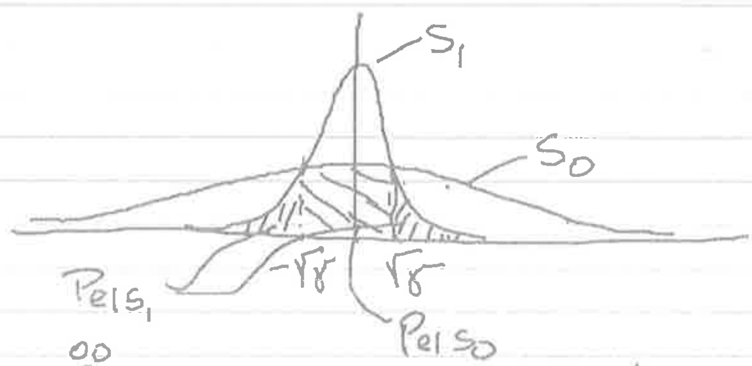
$E[R|S_1] = 0 \qquad Var[R|S_1] = \frac{N_0}{2} \cdot T$

b) $\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0 \end{cases}$



$P_e = P_{e|S_0} = P_{e|S_1} = \frac{1}{2}$

c) $\hat{m} = \begin{cases} 0 & \text{if } R^2 > \gamma \ (|R| > \sqrt{\gamma}) \\ 1 & \text{if } R^2 < \gamma \ (|R| < \sqrt{\gamma}) \end{cases}$



$P_{e|S_1} = 2 \cdot \int_{\sqrt{\gamma}}^{\infty} P_{R|S_1}(r)\, dr = 2Q\left(\frac{\sqrt{\gamma}}{\sqrt{\frac{N_0}{2} \cdot T}}\right) = 2 \cdot Q\left(\sqrt{\frac{2\gamma}{N_0 T}}\right)$

$P_{e|S_0} = 1 - 2 \cdot \int_{\sqrt{\gamma}}^{\infty} P_{R|S_0}(r)\, dr = 1 - 2 \cdot Q\left(\sqrt{\frac{2\gamma}{(E+N_0)T}}\right)$

$$P_e = \frac{1}{2}\cdot\left(P_{e|s_0} + P_{e|s_1}\right)$$

$$= \frac{1}{2} - Q\left(\sqrt{\frac{2\gamma}{(E+N_0)T}}\right) + Q\left(\sqrt{\frac{2\gamma}{N_0 T}}\right)$$

with $\gamma = N_0 \cdot T \cdot \ln\left(\frac{E}{N_0}\right)$

$E = 5 \cdot N_0$

$$P_e = \underbrace{\frac{1}{2} - Q\left(\frac{\ln 5}{3}\right)}_{\pi_0 \cdot P_{e|s_0}} + \underbrace{Q\left(2 \ln 5\right)}_{\pi_1 \cdot P_{e|s_1}}$$

$$\approx \frac{1}{2} - 0.296 + 64 \cdot 10^{-4} \approx 0.205$$