HW 6 Solution

① $s_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $s_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $s_2 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

a) Gram - Schmidt procedure:

$$\psi_0 = \frac{s_0}{\|s_0\|} = \frac{s_0}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\tilde{\psi}_1 = s_1 - \underbrace{\langle s_1, \psi_0 \rangle}\cdot \psi_0$$

$$\langle s_1, \psi_0 \rangle = [1\ 2\ 3] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\tilde{\psi}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\cancel{\sqrt{3}} \cdot \frac{1}{\cancel{\sqrt{3}}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\psi_1 = \frac{\tilde{\psi}_1}{\|\tilde{\psi}_1\|} = \frac{\tilde{\psi}_1}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\tilde{\psi}_2 = s_2 - \langle s_2, \psi_0 \rangle \cdot \psi_0 - \langle s_2, \psi_1 \rangle \cdot \psi_1$$

$$\langle s_2, \psi_0 \rangle = [1\ 4\ 9] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} = \frac{14}{\sqrt{3}}$$

$$\langle s_2, \psi_1 \rangle = [1\ 4\ 9] \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

$$\tilde{\psi}_2 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} - \frac{14}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{8}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} \cdot$$

$$\boxed{\psi_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

b) Dimension = 3

c) $s_i = \sum_{n=0}^{2} \langle s_i, \psi_n \rangle \cdot \psi_n$

$$s_0 = \sqrt{3} \cdot \psi_0$$

$$s_1 = 2\sqrt{3} \cdot \psi_0 + \sqrt{2} \cdot \psi_1$$

$$s_2 = \frac{14}{\sqrt{3}} \cdot \psi_0 + \frac{8}{\sqrt{2}} \cdot \psi_1 + \frac{2}{\sqrt{6}} \cdot \psi_2$$

d)
$$\psi_0(t) = \frac{1}{\sqrt{3}} \cdot \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & \text{else} \end{cases} \qquad \psi_1(t) = \frac{1}{\sqrt{2}} \cdot \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & \text{else} \end{cases}$$

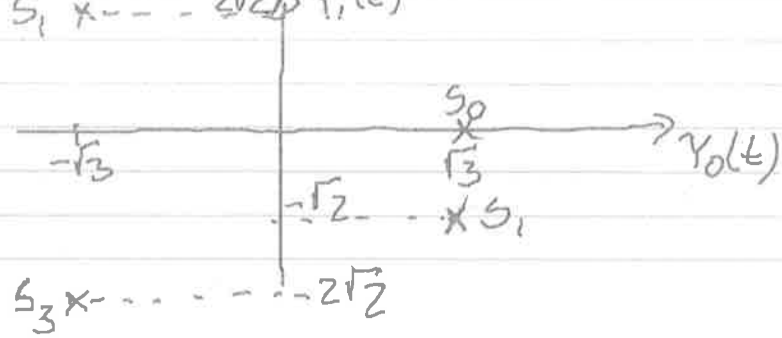
Dimension: 2

$$s_0(t) = \sqrt{3} \cdot \psi_0(t)$$

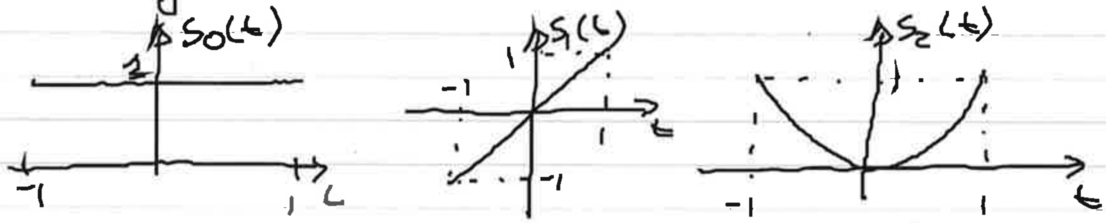
$$s_1(t) = -\sqrt{3} \cdot \psi_0(t) + 2\sqrt{2} \cdot \psi_1(t)$$

$$s_2(t) = \sqrt{3} \cdot \psi_0(t) - \sqrt{2} \cdot \psi_1(t)$$

$$s_3(t) = -\sqrt{3} \cdot \psi_0(t) - 2\sqrt{2} \cdot \psi_1(t)$$



② 3 signals



$s_0(t) = 1$ $\qquad$ $s_1(t) = t$ $\qquad$ $s_2(t) = t^2$

a) Gram-Schmidt:

0. $e_0(t) = \frac{s_0(t)}{\|s_0(t)\|} = \frac{s_0(t)}{\sqrt{2}} = \underline{\frac{\sqrt{2}}{2}}$ for $-1 \le t \le 1$

1. a) $\tilde{e}_1(t) = s_1(t) - \langle s_0(t), e_0(t)\rangle \cdot e_0(t)$

$\langle s_1(t), e_0(t)\rangle = \int_{-1}^{1} t \cdot \frac{\sqrt{2}}{2}\, dt = \frac{t^2}{2} \cdot \frac{\sqrt{2}}{2}\Big|_{-1}^{1} = 0$

$\tilde{e}_1(t) = s_1(t) = t$ for $-1 \le t \le 1$

b) $e_1(t) = \frac{\tilde{e}_1(t)}{\|\tilde{e}_1(t)\|} = \underline{\sqrt{\frac{3}{2}} \cdot t}$ for $-1 \le t \le 1$

2. a) $\tilde{e}_2(t) = \underline{s_2(t)} - \langle s_2(t), e_0(t)\rangle e_0(t) - \langle s_2(t), e_1(t)\rangle e_1(t)$

$\langle s_2(t), e_0(t)\rangle = \int_{-1}^{1} t^2 \cdot \frac{\sqrt{2}}{2}\, dt = \frac{\sqrt{2}}{3}$

$\boxed{\|e_2(t)\|^2 = \frac{8}{45}}$ $\qquad$ $\langle s_2(t), e_1(t)\rangle = \int_{-1}^{1} t^2 \cdot \sqrt{\frac{3}{2}} \cdot t\, dt = 0$

$\tilde{e}_2 = s_2 - \frac{\sqrt{2}}{3} \cdot \frac{\sqrt{2}}{2} = t^2 - \frac{1}{3}$ $\quad$ $-1 \le t \le 1$

b) $e_2(t) = \frac{\tilde{e}_2(t)}{\|\tilde{e}_2(t)\|} = \underline{\sqrt{\frac{5}{2}} \cdot \frac{1}{2} \cdot (3t^2 - 1)}$ $\quad$ $-1 \le t \le 1$

b) $P_n(t) = \frac{(-1)^n}{2^n n!} \cdot \frac{d^n(1-t^2)^n}{dt^n}$ $\qquad e_n(t) = \sqrt{\frac{2n+1}{2}} \cdot P_n(t)$

$$P_0 = \frac{1}{1\cdot 1}\,\frac{d^0(1-t^2)^0}{dt^0} = 1$$

$$P_1 = \frac{-1}{2\cdot 1}\cdot\frac{d(1-t^2)^1}{dt} = t$$

$$P_2 = \frac{1}{4\cdot 2}\cdot\frac{d^2(1-t^2)^2}{dt^2} = \frac{1}{2}(3t^2-1)$$

$$e_0(t) = \sqrt{\frac{2\cdot 0+1}{2}}\cdot P_0(t) = \sqrt{\frac{1}{2}}\cdot 1 = \underline{\underline{\frac{\sqrt{2}}{2}}} \quad -1\le t\le 1$$

$$e_1(t) = \sqrt{\frac{2\cdot 1+1}{2}}\cdot P_1(t) = \sqrt{\frac{3}{2}}\cdot t \quad -1\le t\le 1$$

$$e_2(t) = \sqrt{\frac{2\cdot 2+1}{2}}\cdot P_2(t) = \sqrt{\frac{5}{2}}\cdot\frac{1}{2}(3t^2-1) \quad -1\le t\le 1$$

c) $x(t) = \cosh(t) = \frac{e^t+e^{-t}}{2}$

$$\hat{x}(t) = \sum_{n=0}^{2} x_n\cdot e_n(t) \quad \text{with } x_n = \underline{\underline{\langle x(t), e_n(t)\rangle}}$$

$$x_0 = \langle x(t), e_0(t)\rangle = \int_{-1}^{1}\cosh(t)\cdot\frac{\sqrt{2}}{2}\,dt = \sqrt{2}\cdot\sinh(1) \approx 1.66$$

$$x_1 = \langle x(t), e_1(t)\rangle = \int_{-1}^{1}\cosh(t)\cdot\sqrt{\frac{3}{2}}\,t\,dt = 0$$

$$x_2 = \langle x(t), e_2(t)\rangle = \int_{-1}^{1}\cosh(t)\cdot\sqrt{\frac{5}{2}}\cdot\frac{1}{2}(3t^2-1)\,dt = \frac{\sqrt{\frac{5}{2}}\cdot(e^2-7)}{e} \approx 0.22$$

$x(t)\approx\hat{x}$

$$\Rightarrow \hat{x}(t) = x_0\cdot e_0(t) + x_2\cdot e_2(t) \approx 0.53t^2 + 0.99$$

(3) $K_x(\tau) = \begin{cases} 1-|\tau| & \text{for } |\tau|<1 \\ 0 & \text{else} \end{cases}$

Karhunen-Loeve expansion:

$$\int_0^T K_x(t-u)\,\phi(u)\,du = \lambda\cdot\phi(t)$$

$$\Rightarrow \int_0^T (1-|t-u|)\,\phi(u)\,du = \lambda\cdot\phi(t)$$

(*) $$\Rightarrow \int_0^t (1-(t-u))\,\phi(u)\,du + \int_t^T (1+(t-u))\,\phi(u)\,du = \lambda\,\phi(t)$$

$$\Rightarrow \int_0^t (1+u)\,\phi(u)\,du - t\int_0^t \phi(u)\,du + \int_t^T (1-u)\,\phi(u)\,du + t\int_t^T \phi(u)\,du = \lambda\phi(t)$$

$\frac{d}{dt}$:

$$\cancel{(1+t)\cdot\phi(t)} - 1\cdot\int_0^t \phi(u)\,du - \cancel{t\cdot\phi(t)} + (-\cancel{(1-t)\cdot\phi(t)}) + 1\cdot\int_t^T \phi(u)\,du + \cancel{t\cdot(-\phi(t))} = \lambda\cdot\phi'(t)$$

$$\Rightarrow -\int_0^t \phi(u)\,du + \int_t^T \phi(u)\,du = \lambda\cdot\phi'(t)$$

$\frac{d^2}{dt^2}$: $-\phi(t) - \phi(t) = \lambda\,\phi''(t)$

$$-2\phi(t) = \lambda\,\phi''(t)$$

$$\boxed{\phi(t) = A\cdot\sin(\omega t) + B\cdot\cos(\omega t)}$$

$\lambda\cdot\phi''(t) = \lambda\omega^2\cdot\phi(t)$

$\cancel{\lambda\neq 0}$ $\Rightarrow \lambda\cdot\omega^2 = 2 \Rightarrow \omega = \sqrt{\frac{2}{\lambda}}$

$$\int_0^t (1-(t-u)) \cdot (A\sin(\omega u) + B\cos(\omega u))\, du + \int_t^T (1+(t+u)) \cdot (A\sin(\omega u) + B\cos(\omega u))\, du \overset{!}{=} \lambda \phi(t)$$

$$\Rightarrow \frac{1}{\omega^2} \cdot \Big[A\omega - A\omega \cdot \cos(\omega T) + A\omega T\cos(\omega T) - A\sin(\omega T) - B - B\cos(\omega T) + B\omega \cdot \sin(\omega T) - B\omega T\sin(\omega T)\Big] +$$

$$\Rightarrow \frac{t}{\omega} \cdot \Big[B \cdot \sin(\omega T) - A - A \cdot \cos(\omega T)\Big] +$$

$$\underbrace{\frac{2}{\omega^2}}_{=\lambda} \cdot \underbrace{\Big[A \cdot \sin(\omega t) + B \cdot \cos(\omega t)\Big]}_{=\phi(t)}$$

A, B, ω must solve:

1.) $\Big[A\omega - A\omega\cos(\omega T) + A\omega T\cos\omega T - A\sin(\omega T) \underbrace{- B - B\cos(\omega T)} + B\omega\sin(\omega T) - B\omega T\sin(\omega T)\Big] \overset{!}{=} 0$

2.) $\Big[B \cdot \sin(\omega T) - A - A\cos(\omega T)\Big] \overset{!}{=} 0$

This is solved by

1.) $A = 0$

2.) $\cos(\omega T) = -1$

$\Rightarrow \sin(\omega T) = 0$

$\Rightarrow \omega_n \cdot T = (2n+1) \cdot \pi$

$\omega_n = \frac{1}{T} \cdot (2n+1) \cdot \pi$

$$\Rightarrow \boxed{\lambda_n = \frac{2}{\omega^2} = \frac{2 \cdot T^2}{\left((2n+1)\cdot\pi\right)^2}}$$

$$\phi_n(t) = B \cdot \cos\left((2n+1)\cdot\pi\cdot\frac{t}{T}\right)$$

$$= \boxed{\sqrt{\frac{2}{T}}\, \cos\left((2n+1)\cdot\pi\cdot\frac{t}{T}\right)}$$

for $\|\phi_n(t)\| = 1$