ECE 630 HW 5 Solution Dr. Paris

1. Four conditions must hold:

i) $\langle x, y\rangle = \langle y, x\rangle$ Symmetry

ii) $\langle ax, y\rangle = a\langle x, y\rangle$

iii) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ } linearity

iv) $\langle x, x\rangle = \|x\|^2 > 0$ unless $x = 0$, then $\langle x, x\rangle = 0$

a) (i) $\langle x, y\rangle = x^T A y$ $x, y \in \mathbb{R}^N$, A $N \times N$ matrix non-singular

(i) does not hold

$\langle x, y\rangle = x^T A y = y^T A^T x \neq \langle y, x\rangle$

unless $A = A^T$

(iv) does not hold unless A is positive definite.

non-singular: eigenvalues $\neq 0$

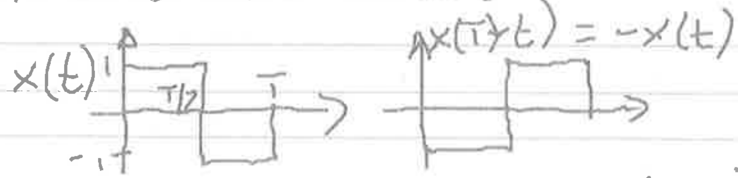
positive definite: eigenvalues > 0

(ii) $\langle x, y\rangle = x \circ y^T$ $x, y \in \mathbb{R}^N$

not inner product, $x \circ y^T$ is $N \times N$ matrix (not scalar)

(iii) $\langle x, y\rangle = \int_0^T x(t) \cdot y(T-t)\, dt$ $x(t), y(t) \in L_2(0,T)$

(iv) does not hold:



$\langle x, x\rangle = \int x(t) \cdot x(T-t)\, dt = -T < 0$

(iv) $\langle x,y\rangle = \int_0^T x(t)\cdot w(t)\cdot y(t)\,dt$

$x(t), y(t) \in L_2(0,T)$

$w(t) \geq 0$

For condition (iv) need.

Counter-example: $w(t) \geq 0$

$$x(t) = \begin{cases} 0 & w(t) > 0 \\ 1 & w(t) = 0 \end{cases}$$

$\langle x(t), x(t)\rangle = 0$ even though $x(t) \neq 0$

(v) $\langle x, y\rangle = E[XY]$ is valid inner product

(vi) $Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$

condition (iv) does not hold for random variables with zero variance (e.g.: $Pr\{X=a\}=1$) $a \neq 0$. $\leftarrow$

Then $Cov(X,\cancel{X}) = Var(X) \cancel{- E^2[X]}$

$= E[X^2] - E^2[X]$

$= a^2 - a^2 = 0$

even though $X \neq 0$

b) Under what condition is

$$\int_0^T \int_0^T x(t)\, Q(t,u)\, y(u)\, du\, dt$$

a valid inner product?

(i) $\langle x,y\rangle = \int_0^T \int_0^T x(\underline{t})\, Q(t,u)\, y(\underline{u})\, du\, dt$

$\langle y,x\rangle = \int_0^T \int_0^T y(\underline{t})\, Q(t,u)\cdot x(\underline{u})\, du\, dt$

$\Rightarrow$ need $Q(t,u) = Q(u,t)$

(ii) $\langle ax, y\rangle = \iint a\,x(t)\,Q(t,u)\,y(u)\,du\,dt$

$= a \iint x(t)\,Q(t,u)\,y(u)\,du\,dt$

$= a\,\langle x, y\rangle$

(iii) $\langle x, y+z\rangle = \iint x(t)\,Q(t,u)\cdot\big(y(u)+z(u)\big)\,du\,dt$

$= \iint x(t)\,Q(t,u)\,y(u)\,du\,dt +$

$\iint x(t)\,Q(t,u)\,z(u)\,du\,dt$

$= \langle x, y\rangle + \langle x, z\rangle$

(iv) $\langle x, x\rangle = \iint x(t)\,\underline{\underline{Q(t,u)}}\,x(u)\,dt\,du > 0$

$\Rightarrow Q(t,u)$ must be positive definite

(like an autocovariance function)

2. $\langle x, y\rangle = \int x(t)\cdot y^*(t)\,dt$

$\mathcal{L}: \{y_n(t) : y_n(t) = X_n \cdot e^{j2\pi nt/T}\}$

Find $\hat{y}_n(t) = \arg\min_{y_n(t)} \|x(t) - y_n(t)\|^2$

a) $\|x(t) - y_n(t)\|^2 = \|x(t) - X_n \cdot \exp(j2\pi nt/T)\|^2$

$$= \langle x(t) - X_n e^{j2\pi nt/T}, x(t) - X_n e^{j2\pi nt/T}\rangle$$

$$= \|x(t)\|^2 - \langle x(t), X_n e^{j2\pi nt/T}\rangle - \langle X_n e^{j2\pi nt/T}, x(t)\rangle + \|X_n e^{j2\pi nt/T}\|^2$$

$$= \|x(t)\|^2 - \int x(t)\cdot X_n^* \cdot e^{-j2\pi nt/T}dt - \int X_n e^{j2\pi nt/T}\cdot x^*(t)\,dt + \int X_n \cdot \cancel{e^{j2\pi nt/T}} \cdot X_n^* \cancel{e^{-j2\pi nt/T}}dt$$

$$= \|x(t)\|^2 - X_n^* \cdot \langle x(t), e^{j2\pi nt/T}\rangle - X_n \cdot \langle e^{j2\pi nt/T}, x(t)\rangle + |X_n|^2 \cdot T$$

$a = \langle x(t), e^{j2\pi nt/T}\rangle$:

$$= \|x(t)\|^2 - \underline{X_n^* \cdot a} - \underline{X_n \cdot a^*} + T\cdot |X_n|^2$$

$$= \|x(t)\|^2 - 2\cdot \mathrm{Re}\{X_n \cdot a^*\} + T\cdot |X_n|^2$$

$\frac{d}{dX_n}$ is not trivial RC calculus.

$\left.\begin{aligned} X_n &= A_n + jB_n \\ a &= a_R + j a_I \end{aligned}\right\}$: $\|x(t)\|^2 - 2\cdot \mathrm{Re}\{(A_n + jB_n)\cdot(a_R - j a_I)\} + T\cdot(A_n^2 + B_n^2)$

$$= \|x(t)\|^2 - 2\cdot(A_n\cdot a_R + B_n a_I) + T\cdot(A_n^2 + B_n^2)$$

$$\frac{d}{dA_n}: \quad -2a_R + 2TA_n \stackrel{!}{=} 0 \Rightarrow A_n = \frac{a_R}{T}$$

$$\frac{d}{dB_n}: \quad -2a_I + 2TB_n \stackrel{!}{=} 0 \Rightarrow B_n = \frac{a_I}{T}$$

$$\hat{X}_n = A_n + jB_n = \frac{a_R}{T} + j\frac{a_I}{T} = \frac{a}{T} = \boxed{\underbrace{\frac{\langle x(t), e^{j2\pi nt/T}\rangle}{T}}}$$

$$\hat{y}_n(t) = \underbrace{\frac{\langle x(t), e^{j2\pi nt/T}\rangle}{T}}_{\hat{X}_n} \cdot e^{j2\pi nt/T}$$

b) $z(t) = x(t) - \hat{y}_n(t) \quad \perp \text{ all } y_n(t) \in \mathcal{L}$

$$\langle y_n(t), z(t)\rangle = \langle y_n(t), x(t)\rangle - \langle y_n(t), \hat{y}_n(t)\rangle$$

$y_n(t) = X_n e^{j2\pi nt/T}$: $\quad X_n \underbrace{\langle e^{j2\pi nt/T}, x(t)\rangle} -$

$$X_n\cdot\langle e^{j2\pi nt/T}, \hat{X}_n \cdot e^{j2\pi nt/T}\rangle$$

$\frac{\langle x(t), e^{j2\pi nt/T}\rangle}{T} = \hat{X}_n$: $\quad = X_n\cdot \hat{X}_n^*\cdot T \quad -$

$\frac{\langle e^{j2\pi nt/T}, x(t)\rangle}{T} = \hat{X}_n^*$: $\quad X_n\cdot\hat{X}_n^*\cdot\underbrace{\langle e^{j2\pi nt/T}, e^{j2\pi nt/T}\rangle}_{=T}$

$$= \underline{\underline{0}}$$

(3) minimize

$$\|\underline{Y} - (a\underline{X} + b\underline{1})\|^2 \quad \text{w.r.t } a, b$$

$$\|Y - (a\underline{X} + b\underline{1})\|^2 = \|Y\|^2 + a^2\|\underline{X}\|^2 + b^2\|\underline{1}\|^2 - 2a\langle \underline{X}, \underline{Y}\rangle - 2b\langle \underline{Y}, \underline{1}\rangle + 2ab\langle \underline{X}, \underline{1}\rangle$$

$$\left.\begin{aligned} \frac{d}{da}: \quad & 2a \cdot \|X\|^2 - 2\langle \underline{X}, \underline{Y}\rangle + 2b\langle x, \underline{1}\rangle \stackrel{!}{=} 0 \\ \frac{d}{db}: \quad & 2b \cdot \|1\|^2 - 2\langle Y, \underline{1}\rangle + 2a\langle x, \underline{1}\rangle \stackrel{!}{=} 0 \end{aligned}\right\}$$

Note: $\cdot \|\underline{1}\|^2 = \sum_{n=1}^{N} 1 = N$

$$\langle X, \underline{1}\rangle = \sum_{n=1}^{N} x_n \qquad \langle Y, \underline{1}\rangle = \sum_{n=1}^{N} Y_n$$

System of equation

$$\underbrace{\begin{pmatrix} \|\underline{X}\|^2 & \langle \underline{x}, \underline{1}\rangle \\ \langle \underline{x}, \underline{1}\rangle & \|1\|^2 \end{pmatrix}}_{K} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} \langle \underline{x}, \underline{Y}\rangle \\ \langle Y, \underline{1}\rangle \end{pmatrix}}_{\underline{r}}$$

$$\Rightarrow \boxed{\begin{pmatrix} a \\ b \end{pmatrix} = K^{-1} \cdot \underline{r}}$$

b) $E\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = ?$ $\quad E\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = E[K^{-1} \cdot \underline{r}]$

$$= K^{-1} \cdot E[\underline{r}]$$

$E[\underline{Y}] = a\underline{X} + b\underline{1}$: $\quad = K^{-1} \cdot \begin{pmatrix} \langle \underline{x}, a\underline{x} + b\underline{1}\rangle \\ \langle \underline{1}, a\underline{x} + b\underline{1}\rangle \end{pmatrix}$

$$= K^{-1} \cdot \begin{pmatrix} a \cdot \|\underline{x}\|^2 + b \cdot \langle \underline{x}, \underline{1} \rangle \\ a \cdot \langle \underline{x}, \underline{1} \rangle + \underline{b}\, \|\underline{1}\|^2 \end{pmatrix}$$

$$= K^{-1} \cdot \underbrace{\begin{pmatrix} \|\underline{x}\|^2 & \langle \underline{x}, \underline{1} \rangle \\ \langle \underline{x}, 1 \rangle & \|\underline{1}\|^2 \end{pmatrix}}_{= K} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \underline{\text{unbiased estimate}}$$

c)

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x = 0:4;
y = [1.3, 0.2, 0.1, -0.4, -1.2];
K = [x*x'   sum(x);
     sum(x) length(x)];
r = [x*y'; sum(y)];
ab = inv(K)*r
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$$\boxed{\begin{array}{l} a = -0.56 \\ b = 1.12 \end{array}}$$

d) $\hat{a} = \langle \underline{y}, \underline{x} \rangle$? $\quad b = \langle \underline{y}, \underline{1} \rangle$?

No! because $\underline{x}$ and $\underline{1}$ are not orthonormal!