ECE 630 HW4 Solution Dr. Paris

1. $$X_t \longrightarrow \boxed{H(f)} \longrightarrow Y_t$$

Given: $$H(f) = \frac{j2\pi f + a}{j2\pi f + 2a}$$

$$K_Y(\tau) = a^2 e^{-2a|\tau|} \qquad m_Y = \frac{1}{2}$$

a)
$$R_Y(\tau) = K_Y(\tau) + m_Y^2$$
$$= a^2 e^{-2a|\tau|} + \frac{1}{4}$$
$$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) \cdot e^{-j2\pi f\tau}\, d\tau$$
$$= \int_{-\infty}^{\infty} \left(\frac{1}{4} + a^2 \underline{e^{-2a|\tau|}} \right) e^{-j2\pi f\tau}\, d\tau$$
$$= \underline{\frac{1}{4}\delta(f) + \frac{4a^3}{4a^2 + 4\pi^2 f^2}}$$

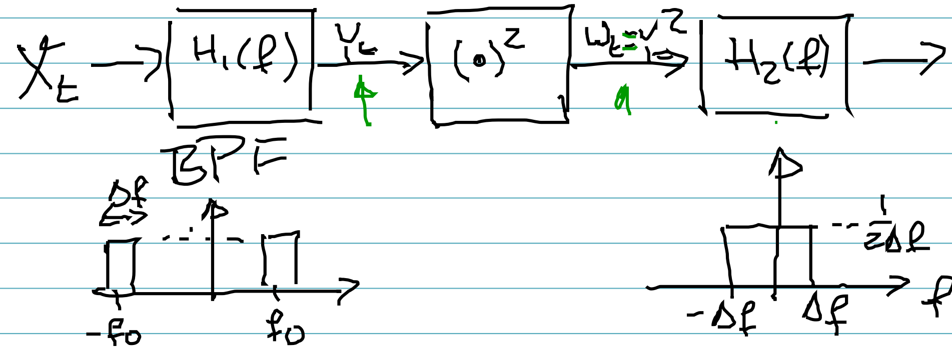
b)
$$S_Y(f) = S_X(f) \cdot |H(f)|^2 \qquad m_Y = H(0) \cdot m_X$$
$$\Rightarrow S_X(f) = \frac{S_Y(f)}{|H(f)|^2} \qquad \Rightarrow m_X = \frac{m_Y}{H(0)}$$
$$|H(f)|^2 = \frac{4\pi^2 f^2 + a^2}{4\pi^2 f^2 + 4a^2} \qquad H(0) = \frac{1}{2}$$
$$S_X(f) = \frac{1/4\,\delta(f)}{|H(0)|^2} + \frac{\dfrac{4a^3}{\cancel{4a^2 + 4\pi^2 f^2}}}{\dfrac{4\pi^2 f^2 + a^2}{\cancel{4\pi^2 f^2 + 4a^2}}} \qquad m_X = \frac{1/2}{1/2} = 1$$
$$= \frac{1/4}{|1/2|^2}\delta(f) + \frac{4a^3}{a^2 + 4\pi^2 f^2}$$

$$R_x(\tau) = 1 + 2a^2 \cdot e^{-a|\tau|}$$

$$K_x(\tau) = 2a^2 e^{-a|\tau|}$$

$$m_x = 1$$

2.

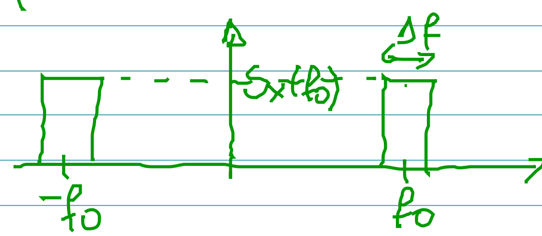


$S_x(f) \triangleq$ PSD of X_t

a)

$$S_Y(f) \approx \begin{cases} S_x(f_0) & f_0 - \frac{\Delta f}{2} \le |f| \le f_0 + \frac{\Delta f}{2} \\ 0 & \text{else} \end{cases}$$

$\underline{S_x(-f_0) = S_x(f_0)}$



$$R_Y(\tau) = \int_{-\infty}^{\infty} S_Y(f) \cdot e^{j2\pi f\tau} \, df$$

$$= \int_{-f_0 - \frac{\Delta f}{2}}^{-f_0 + \Delta f/2} \underbrace{S_x(f_0)} \cdot e^{j2\pi f\tau} df + \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} \underline{S_x(f_0)} \, e^{j2\pi f\tau} df$$

$$= S_x(f_0) \cdot 2 \cdot \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} \cos(2\pi f t) df$$

$$= S_x(f_0) \cdot 2\Delta f \cdot \cos(2\pi f_0 \tau) \cdot \text{sinc}(\pi \Delta f \tau)$$

$W_t = Y_t^2$ second order description

$$E[W_t] = E[Y_t^2] = R_Y(0) = \underline{S_X(f_0) \cdot 2\Delta f}$$

$$R_W(\tau) = E[W_t \cdot W_{t+\tau}] = E[Y_t^2 \cdot Y_{t+\tau}^2]$$

Note that Y_t and $Y_{t+\tau}$ are jointly Gaussian noise

Isserlin's theorem: X, Y jointly Gaussian

then $E[X^2Y^2] = E[X^2]\,E[Y^2] + 2E^2[XY]$

$$E[Y_t^2 \cdot Y_{t+\tau}^2] = E[Y_t^2] \cdot E[Y_{t+\tau}^2] + 2E^2[Y_t \cdot Y_{t+\tau}]$$

$$= R_Y(0) \cdot R_Y(0) + 2R_Y^2(\tau)$$

$$= (\underline{S_X(f_0) \cdot 2\Delta f})^2 + 2 \cdot (S_X(f_0) \cdot 2\Delta f \cdot \underline{\cos(2\pi f_0 \tau)} \cdot \underline{\mathrm{sinc}(\pi \Delta f \tau)})^2$$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

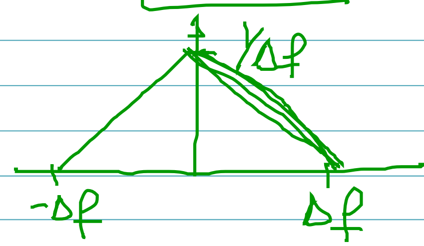
$$= \boxed{(S_X(f_0) \cdot 2\Delta f)^2 \cdot (1 + \mathrm{sinc}^2(\pi \Delta f \tau)) + (S_X(f_0)\, 2\Delta f)^2 \cdot (\underline{\cos(4\pi f_0 \tau)} \cdot \mathrm{sinc}^2(\pi \Delta f \tau))}$$

rejected by LPF

b) $R_Z(\tau) = \left(\frac{1}{2\Delta f}\right)^2 \cdot \left(S_X(f_0)\cdot 2\Delta f\right)^2 \cdot \left(1 + \text{sinc}^2(\pi \Delta f \tau)\right)$

$$= \underline{S_X^2(f_0)} \cdot \left(\underline{1} + \underline{\text{sinc}^2(\pi\Delta f \tau)}\right)$$

$$S_Z(f) = S_X^2(f_0)\cdot\left(\delta(f) + \underline{\frac{1}{\Delta f}\cdot\left(1 - \frac{|f|}{\Delta f}\right)}\right) \quad \text{for } |f|<\Delta f$$



c) $E[Z_t] = H(0)\cdot E[Y_t^2]$

$$= H(0)\cdot E[W_t]$$

$$= \cancel{\frac{1}{2\Delta f}} \cdot S_X(f_0)\cdot \cancel{2\Delta f} = \underline{\underline{S_X(f_0)}}$$

<u>Good!</u> Unbiased estimate of $S_X(f_0)$

d) $\text{Var}[Z_t] = R_Z(0) - m_Z^2$

$$= S_X^2(f_0)\cdot(1+1) - S_X^2(f_0)$$

$$= S_X^2(f_0)$$

<u>Bad!</u> Standard deviation equals $S_X(f_0)$. As large as $S_X(f_0)$.

3. $$X_t = \int_0^t W_s \, ds$$

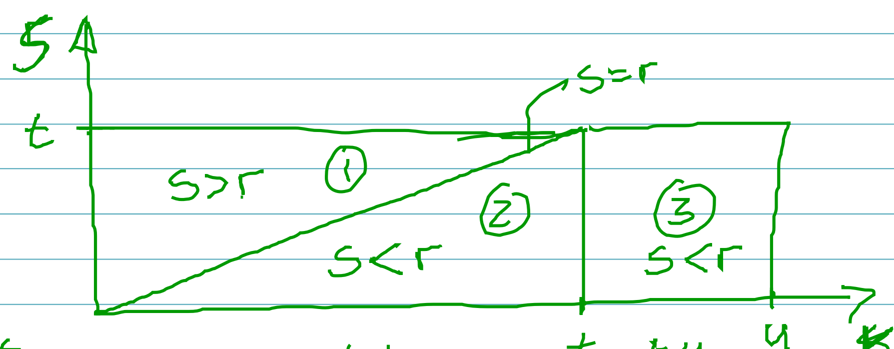
W_t — Wiener Process

Recall: $E[W_t] = 0$ $R_W(t,u) = \sigma^2 \min(t,u)$

a) $$E[X_t] = E\left[\int_0^t W_s \, ds\right] = \int_0^t E[W_s] \, ds = \underline{\underline{0}}$$

b) $$R_X(t,u) = E[X_t \cdot X_u] = E\left[\int_0^t W_s \, ds \cdot \int_0^u W_r \, dr\right] = \int_0^t \int_0^u E[W_s W_r] \, ds \, dr = \int_0^t \int_0^u \sigma^2 \min(s,r) \, ds \, dr$$

Assume first that $u \geq t$:



$$\underbrace{\sigma^2 \int_0^t \int_0^s r \, dr \, ds}_{(1)} + \underbrace{\sigma^2 \int_0^t \int_s^t s \, dr \, ds}_{(2)} + \underbrace{\sigma^2 \int_0^t \int_t^u s \, dr \, ds}_{(3)}$$

(1) $$\int_0^t \int_0^s r \, dr \, ds = \int_0^t \frac{r^2}{2}\Big|_0^s ds = \int_0^t \frac{s^2}{2}$$
$$= \cancel{\int_0} \frac{s^3}{6}\Big|_0^t = \frac{t^3}{6}$$

(2) $$\int_0^t \int_s^t s \, dr \, ds = \int_0^t rs \Big|_s^t ds = \int_0^t (ts - s^2) ds$$
$$= \frac{ts^2}{2} - \frac{s^3}{3}\Big|_0^t = \frac{t^3}{6}$$

(3) $$\int_0^t \int_t^u s \, dr \, ds = \int_0^t r \cdot s \Big|_t^u ds = \int_0^t (u-t) \cdot s \, ds$$
$$= (u-t) \cdot \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}(u-t)$$

For $u \geq t$:
$$R_x(t,u) = \sigma^2 \cdot \frac{t^2 u}{2} - \frac{\sigma^2 \cdot t^3}{6}$$

Analogously, for $t \geq u$:
$$R_x(t,u) = \sigma^2 \cdot \frac{u^2 t}{2} - \frac{\sigma^2 u^3}{6}$$

Combined:
$$\boxed{R_x(t,u) = \sigma^2 \cdot u \cdot t \cdot \min(t,u) - \frac{\sigma^2}{6}(\min(t,u))^3}$$

c) X_t is not WSS

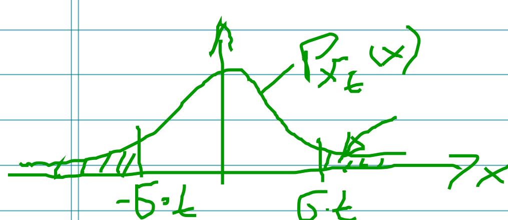
$$R_X(t,u) \neq R_X(t-u)$$

d) $\Pr\{|X_t| > \sigma \cdot t\}$

pdf of X_t ?

- X_t is Gaussian
  - linear operation on white Gaussian noise
- Need: $E[X_t] = 0$

$$\mathrm{Var}[X_t] = E[X_t \cdot X_t]$$
$$= R_X(t,t)$$
$$= \sigma^2 \cdot \frac{t^3}{3}$$



$$\Pr\{|X_t| > \sigma \cdot t\} = 2 \cdot \Pr\{X_t > \sigma t\}$$

$$= 2 \cdot \int_{\sigma t}^{\infty} \frac{1}{\sqrt{2\pi \sigma^2 \frac{t^3}{3}}} \cdot \exp\left(-\frac{1}{2} \frac{x^2}{\sigma^2 \frac{t^3}{3}}\right) dx$$

$$z = \frac{x - 0}{\sqrt{\sigma^2 t^3 / 3}}$$

$$= 2 \cdot \int_{1/\sqrt{t/3}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

Lower: $\frac{\cancel{\sigma \cdot t}}{\sqrt{\cancel{\sigma^2} \cancel{t^2} \cdot t/3}}$

$$= 2 \cdot Q\left(\sqrt{3/t}\right)$$