ECE 630 HW2 - Solution Dr Paris

1. $\underline{X} \sim N\left(\underline{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}\right)$

a) $Y_1 = X_1 + 2X_2$

$Y_2 = -X_1 + X_2$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}}_{A} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$C_Y = A \cdot C \cdot A^T = \begin{pmatrix} 13 & 8 \\ 8 & 7 \end{pmatrix}$$

$$Cov(Y_1, Y_2) = 8$$

b) $m_Y = A \cdot \underline{m}_X = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$$f_{\underline{Y}}(\underline{y}) = \frac{1}{2\pi \cdot \sqrt{|C_Y|}} \cdot \exp\left(-\frac{1}{2} (\underline{y} - \underline{m}_Y)^T \cdot C_Y^{-1} (\underline{y} - \underline{m}_Y)\right)$$

$|C_Y| = 27$

$C_Y^{-1} = \frac{1}{27} \cdot \begin{pmatrix} 7 & -8 \\ -8 & 13 \end{pmatrix}$

$$= \frac{1}{2\pi\sqrt{27}} \cdot \exp\left(-\frac{1}{2} \frac{7 \cdot (y_1 - 4)^2 - 16 \cdot (y_1 - 4)(y_2 + 1) + 13(y_2+1)^2}{27}\right)$$

c) $\Pr\{Y_1 > 2Y_2 + 1\} = \Pr\{\underbrace{Y_1 - 2Y_2}_{Z = Y_1 - 2Y_2} > 1\} = \Pr\{Z > 1\}$

$Z = \underbrace{(1 \; -2)}_{b} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$m_Z = b \cdot m_Y = 6$

$\sigma_Z^2 = b \cdot C_Y \cdot b^T = 9$

$$= \int_1^{\infty} \frac{1}{\sqrt{2\pi \cdot 9}} \cdot \exp\left(-\frac{1}{2} \frac{(z-6)^2}{9}\right) dz$$

$$= \int_{-\frac{5}{3}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad x = \frac{z-6}{3}$$

$$= Q\left(-\frac{5}{3}\right) = 1 - Q\left(\frac{5}{3}\right)$$

2. X_1, X_2 - independent, identically distributed
- $N(0, \nu^2)$

Transform to polar coordinates (R, Φ)

$$X_1 = \underline{R \cos\Phi} . \qquad \underline{X_2 = R \cdot \sin\Phi} .$$

a) Jacobian Matrix:

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Phi} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Phi} \end{pmatrix} = \begin{pmatrix} \cos\Phi & -R\sin\Phi \\ \sin\Phi & R\cdot\cos\Phi \end{pmatrix}$$

$$\Rightarrow |J| = R\cdot\cos^2\Phi + R\cdot\sin^2\Phi = R$$

$$P_{R,\Phi}(r,\phi) = P_{X_1,X_2}(x_1, x_2) \cdot |J| \Big|_{\substack{x_1 = r\cos\phi \\ x_2 = r\cdot\sin\phi}}$$

$$= \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{1}{2}\left(\frac{(r\cos\phi)^2}{\nu^2} + \frac{(r\sin\phi)^2}{\nu^2}\right)\right) \cdot r$$

$$= \frac{r}{2\pi\nu^2} \cdot \exp\left(-\frac{r^2}{2\nu^2}\right) \qquad \text{for: } r \geq 0, \quad 0 \leq \phi < 2\pi$$

b) $$P_{R,\Phi}(r,\phi) = \underbrace{\frac{r}{\nu^2}\exp\left(-\frac{r^2}{2\nu^2}\right)} \cdot \underbrace{\frac{1}{2\pi}} \qquad r \geq 0, \quad 0 \leq \phi < 2\pi$$

$$= \quad P_R(r) \quad \cdot P_\Phi(\phi)$$

where: $P_R(r) = \frac{r}{\nu^2}\exp\left(-\frac{r^2}{2\nu^2}\right) \qquad r \geq 0$

$P_\Phi(\phi) = \frac{1}{2\pi} \qquad 0 \leq \phi < 2\pi$

$\Rightarrow R, \Phi$ independent

c) $Z = R^2$

$$F_Z(z) = \Pr\{Z \le z\}$$
$$= \Pr\{R^2 \le z\}$$
$$= \Pr\{R \le \sqrt{z}\}$$
$$= \int_{-\infty}^{\sqrt{z}} p_R(r)\,dr$$
$$= \int_0^{\sqrt{z}} \frac{r}{\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\,dr \qquad \Big| \int \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr = -e^{-r^2/2\sigma^2}$$
$$= -\exp\left(-\frac{r^2}{2\sigma^2}\right)\Big|_0^{\sqrt{z}}$$
$$= 1 - \exp\left(-\frac{z}{2\sigma^2}\right) \qquad z \ge 0$$

$\Rightarrow$ pdf: $$\frac{d}{dz}F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2\sigma^2} \cdot \exp\left(-\frac{z}{2\sigma^2}\right) \end{cases}$$

d) $$E[Z] = \int_0^\infty z \cdot \frac{1}{2\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) dz = 2\sigma^2$$

20dB lower than $E[Z]$: $\frac{E[Z]}{100}$

20 dB = 100

$$\Pr\left\{Z \le \frac{E[Z]}{100}\right\} = \Pr\left\{Z \le \frac{2\sigma^2}{100}\right\} = F_Z\left(\frac{2\sigma^2}{100}\right)$$
$$= 1 - \exp\left(-\frac{2\sigma^2/100}{2\sigma^2}\right)$$
$$= 1 - \exp\left(-\frac{1}{100}\right) \approx \frac{1}{100}$$

$e^x \approx 1 + x$

$$X_1 \sim N(m_1, v^2)$$
$$X_2 \sim N(m_2, v^2)$$

using same Jacabian as in a).

$$p_{R,\Phi}(r,\phi) = \frac{r}{2\pi v^2}\cdot \exp\left(-\frac{1}{2v^2}\cdot\left((r\cos\phi - m_1)^2 + (r\sin\phi - m_2)^2\right)\right)$$

There exist, m, θ:

$$m_1 = m\cdot\cos\theta$$
$$m_2 = m\cdot\sin\theta$$

$$p_{R,\Phi}(r,\phi) = \frac{r}{2\pi v^2}\exp\left(-\frac{1}{2v^2}\cdot\left((r\cos\phi - m\cos\theta)^2 + (r\sin\phi - m\sin\theta)^2\right)\right)$$

$$= \frac{r}{2\pi v^2}\cdot\exp\left(-\frac{r^2+m^2}{2v^2}\right)\cdot\exp\left(\frac{rm}{v^2}\cdot\underbrace{(\cos\phi\cdot\cos\theta + \sin\phi\sin\theta)}_{=\cos(\phi-\theta)}\right)$$

$$= \frac{r}{2\pi v^2}\cdot\exp\left(-\frac{r^2+m^2}{2v^2}\right)\cdot\exp\left(\frac{rm}{v^2}\cdot\cos(\phi-\theta)\right)$$

for $r \geq 0$, $0 \leq \phi < 2\pi$

$$p_R(r) = \int_0^{2\pi} p_{R\Phi}(r,\phi)\,d\phi$$

$$= \frac{r}{v^2}\exp\left(-\frac{r^2+m^2}{2v^2}\right)\cdot\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{rm}{v^2}\cdot\cos(\phi-\theta)\right)d\phi}_{=I_0\left(\frac{rm}{v^2}\right)}$$

$$= \frac{r}{v^2}\cdot\exp\left(-\frac{r^2+m^2}{2v^2}\right)\cdot I_0\left(\frac{rm}{v^2}\right)$$

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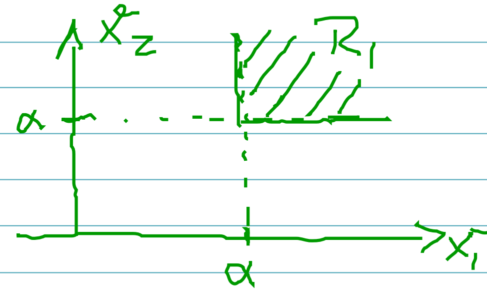
3. X_1, X_2 iid, $N(0,1)$

a)

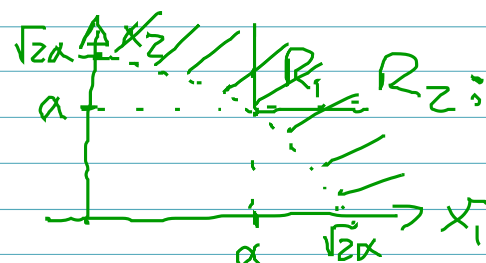


b) $\Pr\{(X_1,X_2)\in R_1\} = \Pr\{X_1>\alpha,\ X_2>\alpha)$

$= \Pr\{X_1>\alpha\}\cdot\Pr\{X_2>\alpha\}$

$= Q(\alpha)\cdot Q(\alpha) = Q^2(\alpha)$

c)



$R_2: x^2+y^2>2\alpha^2,\ x,y\geq 0$

Note: $R_2 \supseteq R_1$

d) $\Pr\{(X_1,X_2)\in R_2\} = \Pr\{X_1^2+X_2^2>2\alpha,\ X_1,X_2\geq 0\}$

$$= \iint\limits_{\substack{x_1^2+x_2^2>2\alpha^2 \\ x_1,x_2\geq 0}} \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)dx_1\,dx_2$$

Polar coordinates:
$r^2 = x^2+y^2$

$$= \int_0^{\pi/2}\int_{\sqrt{2}\alpha}^{\infty} \frac{1}{2\pi}\cdot\exp\left(-\frac{r^2}{2}\right)\underline{r}\cdot dr\,d\phi$$

$$= \frac{1}{4}\cdot\left(-e^{-r^2/2}\right)\Big|_{\sqrt{2}\cdot\alpha}^{\infty}$$

$$= \frac{1}{4}\cdot e^{-\alpha^2}$$

$$\Pr\{(X_1, X_2) \in R_2^2\} \geq \Pr\{(X_1, X_2) \in R_1\}$$

$$\Rightarrow \frac{1}{4} e^{-\alpha^2} \geq Q^2(\alpha)$$

$$\Rightarrow \boxed{\frac{1}{2} e^{-\alpha^2/2} \geq Q(\alpha)}$$

4.

$$\underline{m} = 0 \qquad K = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

a)

$$p_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{|K|}} \cdot \exp\left(-\frac{1}{2}\, \underline{x}^T \cdot K^{-1} \cdot \underline{x}\right)$$

$$|K| = 48 \qquad K^{-1} = \frac{1}{48} \cdot \begin{bmatrix} 40 & 24 & 0 \\ 24 & 24 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

b)

$$Y = X_1 + 2X_2 - X_3$$

$$= \underbrace{\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}}_{a} \cdot \underline{X}$$

$$Y \sim N\left(\underline{a} \cdot \underline{m},\ a \cdot K \cdot a^T\right) = N(0, 19)$$

c)

$$Z_1 = 5X_1 - 3X_2 - X_3$$

$$Z_2 = -X_1 + 3X_2 - X_3$$

$$Z_3 = X_1 + X_3$$

$$\underline{Z} = \underbrace{\begin{bmatrix} 5 & -3 & -1 \\ -1 & +3 & -1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \cdot \underline{X}$$

$$\underline{Z} \sim N\left(A \cdot \underline{m},\ A \cdot K A^T\right)$$

$$\Rightarrow \underline{Z} \sim N(\underline{0}, \Sigma^{Z})$$

$$\Sigma^{Z} = \begin{bmatrix} 218 & -106 & 16 \\ -106 & 74 & -20 \\ 16 & -20 & 11 \end{bmatrix}$$

d) $$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1 X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

i) $$f_{X_1 X_2}(x_1, x_2) = \int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x})\, dx_3$$

$$\cancel{f_X} \quad \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\Big(\underline{0}, \underbrace{\begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}}_{\tilde{K}}\Big)$$

ii) $$f_{X_2}(x_2) \sim N(0, 5)$$

$$f_{X_1|X_2}(x_1|x_2) = \frac{\frac{1}{2\pi \cdot \sqrt{|\tilde{K}|}} \cdot \exp\left(-\frac{1}{2}[x_1\ x_2] \cdot \tilde{K}^{-1} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{\frac{1}{\sqrt{2\pi \cdot 5}} \cdot \exp\left(-\frac{1}{2} \frac{x_2^2}{5}\right)}$$

$|\tilde{K}| = 6$

$$\tilde{K}^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$$

$$= \frac{\frac{1}{2\pi\sqrt{6}} \cdot \exp\left(-\frac{1}{2}\left(\frac{5}{6}x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2\right)\right)}{\frac{1}{\sqrt{2\pi}\sqrt{5}} \cdot \exp\left(-\frac{1}{2} \frac{x_2^2}{5}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{6/5}} \cdot \exp\left(-\frac{1}{2}\left(\frac{5}{6}x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{5}x_2^2\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{6/5}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{(x_1 + \frac{3}{5}x_2)^2}{6/5}\right)$$

$$= N\left(\tfrac{3}{5}x_2, \ 6/5\right)$$